Let $p$ be a prime, and $q = p^m$, a new equivalence relation $\sim_n$ is introduced on the nonzero elements of the finite field $\mathbb{F}_q$ to classify constacyclic codes of length $n$ over $\mathbb{F}_q$ such that the constacyclic codes belonging to the same isometry class have the same distance structures and the same algebraic structures. Some necessary and sufficient conditions for any two nonzero elements of $\mathbb{F}_q$ to be equivalent to each other are presented. We show that if $\lambda \sim_n \mu$ then there exists a very explicit $\mathbb{F}_q$-algebra isomorphism $\phi$ between $\mathbb{F}_q[X]/\langle X^n - \lambda \rangle$ and $\mathbb{F}_q[X]/\langle X^n - \mu \rangle$.

As an application, we provide all the equivalence classes induced by $\sim_{\ell p^s}$, and characterize all constacyclic codes of length $\ell p^s$ over $\mathbb{F}_{p^m}$ and their duals, where $\ell$ is a prime different from $p$. (Received August 11, 2015)