It is well known that solutions of a linear elliptic equation, for instance a harmonic function $\Delta u = 0$, are automatically smooth. Indeed, one has many ways to quantify this and understand precise estimates on solutions. In the context of nonlinear equations, for instance nonlinear harmonic maps, Yang-Mills, Einstein manifolds, etc..., this need not be the case and one may be forced to deal with a singular set $\text{Sing}(u)$. Hausdorff dimension estimates on the singular set have been understood since the days of Federer in the 60’s, but more refined structure and the ability to produce effective estimates on $u$ have remained elusive.

We will discuss in this talk a new series of techniques for analyzing nonlinear differential equations. The talk will focus on nonlinear harmonic maps, however the techniques have found applications in many equations. We will see that the singular set $\text{Sing}(u)$ of such nonlinear equations have manifold structure, or more precisely a rectifiable structure, and how to use these ideas to produce sharp estimates on solutions. Further refinements of these ideas have been used to solve a variety of open conjectures, including the Energy Identity conjecture for Yang-Mills and the $n - 4$-finiteness conjecture for Einstein manifolds. (Received January 22, 2019)