We compute the Picard group of the category of $K(2)$-local module spectra over the ring spectrum $E^{hC_4}$, where $E$ is a height 2 Morava $E$-theory and $C_4$ is a subgroup of the associated Morava stabilizer group. This group can be identified with the Picard group of $K(2)$-local $E$-modules in genuine $C_4$-spectra. We show that in addition to a cyclic subgroup of order 32 generated by $E \wedge S^1$ the Picard group contains a subgroup of order 2 generated by $E \wedge S^7+\sigma$, where $\sigma$ is the sign representation of the group $C_4$. In the process, we completely compute the $RO(C_4)$-graded Mackey functor homotopy fixed point spectral sequence for the $C_4$-spectrum $E$. (Received January 15, 2019)