Let $X$ be a Kähler manifold, let $E \to X$ be a holomorphic vector bundle and let $\pi : X \to U$ be a (not necessarily proper) holomorphic submersion. Given a Hermitian metric on $E$ and a Kähler metric on $X$, one defines a family of Hilbert spaces of square integrable holomorphic sections over $U$. Under the right circumstances this Hilbert bundle (whose fibers can be finite or infinite dimensional) is locally trivial, and one asks about the positivity of the curvature of the Chern connection defined by the $L^2$ metric. The case of a holomorphic family of line bundles, such theorems were proved by Berndtsson. We state analogous results for the higher rank case. We also discuss some applications, as time permits. (Received July 11, 2019)