

# FINITARY QUADRATIC JORDAN ALGEBRAS

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ABSTRACT. Let  $\mathbb{F}$  be a field of arbitrary characteristic. A quadratic Jordan  $\mathbb{F}$ -algebra  $J$  will be called finitary if  $U_a J$  is finite dimensional for any element  $a \in J$ . We determine the structure of nondegenerate finitary quadratic Jordan  $\mathbb{F}$ -algebras.

## INTRODUCTION

In this paper we deal with *quadratic Jordan algebras* (definition below) over a field  $\mathbb{F}$  of arbitrary characteristic. Any associative  $\mathbb{F}$ -algebra  $A$  gives rise to a Lie algebra  $A^{(-)}$  with Lie product  $[a, b] = ab - ba$ , and a quadratic Jordan algebra  $A^{(+)}$  with quadratic mappings  $a \mapsto a^2$  and  $a \mapsto U_a$ ,  $U_a b = aba$  for all  $a, b \in A$ .

Let  $X$  be a vector space over  $\mathbb{F}$ . Denote by  $\mathcal{F}(X)$  the associative algebra of finite-rank linear mappings of the vector space  $X$ . A Lie  $\mathbb{F}$ -algebra  $L$  is said to be *finitary* if it is isomorphic to a subalgebra of the Lie algebra  $\mathfrak{fgl}(X) := \mathcal{F}(X)^{(-)}$  for some vector space  $X$  over  $\mathbb{F}$ . Infinite-dimensional simple finitary Lie algebras over a field of characteristic 0 were classified by Baranov [2]. Later, Baranov and Strade [3] classified infinite-dimensional simple finitary Lie algebras over an algebraically closed field of characteristic not 2 or 3

By analogy with the Lie case, by a *finitary associative  $\mathbb{F}$ -algebra* we mean any subalgebra  $A$  of  $\mathcal{F}(X)$  for some vector space  $X$  over  $\mathbb{F}$ .

Following the same analogy, we could define a finitary quadratic Jordan  $\mathbb{F}$ -algebra as a subalgebra  $J$  of  $\mathcal{F}(X)^{(+)}$  for some vector space  $X$  over  $\mathbb{F}$ . But such a definition would rule out all exceptional quadratic Jordan algebras. Yet being unsuitable, the above definition still helps us to find the correct one. A quadratic Jordan  $\mathbb{F}$ -algebra  $J$  will be called *finitary* if the inner ideal  $U_a J$  is finite-dimensional for every element  $a \in J$ . As will be seen, any subalgebra of a quadratic Jordan algebra  $\mathcal{F}(X)^{(+)}$  is finitary according to this definition.

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Recall that a quadratic Jordan algebra  $J$  is said to be nondegenerate if  $U_a J = 0$  implies  $a = 0$ . If  $J = A^{(+)}$  for some associative algebra  $A$ , then  $J$  is nondegenerate if and only if  $A$  is semiprime.

The main result of this paper is a structure theorem for nondegenerate finitary quadratic Jordan algebras over a field  $\mathbb{F}$  of arbitrary characteristic. Among other results, we will prove that any infinite dimensional central simple finitary quadratic Jordan algebra comes from a central simple finitary associative algebra (with or without involution), so we begin by studying finitary associative algebras.

## 1. FINITARY ASSOCIATIVE ALGEBRAS

Throughout this section  $\mathbb{F}$  will denote a field and  $A$  an associative  $\mathbb{F}$ -algebra. It is convenient to introduce some notation.

**1.1.** Following [7], let  $(X, Y, \langle \cdot, \cdot \rangle)$  be a pair of dual vector spaces over a division  $\mathbb{F}$ -algebra  $\Delta$ , where  $X$  is a left vector space,  $Y$  is a right vector space and  $\langle \cdot, \cdot \rangle : X \times Y \rightarrow \Delta$  is a nondegenerate bilinear form. A linear mapping  $a : X \rightarrow X$  is *adjointable* if there exists  $a^\# : Y \rightarrow Y$  such that  $\langle xa, y \rangle = \langle x, a^\#y \rangle$  for all  $x \in X, y \in Y$ . Notice that we write the mappings of a left vector space on the right (thus composing them from left to right), and the mappings on a right vector space on the left (thus composing them from right to left). We denote by  $\mathcal{F}_Y(X)$  the associative  $\mathbb{F}$ -algebra of all adjointable linear mappings of  $X$ , and by  $\mathcal{F}_Y(X)$  the ideal of those linear mappings having finite rank. The algebras  $\mathcal{F}_Y(X)$  are precisely those simple algebras containing minimal one-sided ideals [7].

Any left vector space  $X$  over  $\Delta$  gives rise to the canonical pair  $(X, X^*, \langle \cdot, \cdot \rangle)$ , where  $X^*$  stands for the dual of  $X$ . Thus, according to the notation above,  $\mathcal{F}(X) = \mathcal{F}_{X^*}(X)$ .

Let  $(X, Y, \langle \cdot, \cdot \rangle)$  be a pair of dual vector spaces over  $\Delta$ . For  $x \in X, y \in Y$ , write  $y \otimes x$  to denote the adjointable linear mapping defined by

$$x'(y \otimes x) = \langle x', y \rangle x \text{ for } x' \in X$$

with adjoint  $(y \otimes x)^\#y' = y \langle x, y' \rangle$ .

The following two results can be easily verified. .

**1.2.**  $(y \otimes x)a = y \otimes xa$  for all mapping  $a : X \rightarrow X$  and  $a(y \otimes x) = a^\#y \otimes x$  for all adjointable  $a$ .

**1.3.** Every  $a \in \mathcal{F}_Y(X)$  can be expressed as  $a = \sum y_i \otimes x_i$  where both sets  $\{y_i\} \subset Y$  and  $\{x_i\} \subset X$  are linearly independent, which just means that  $\mathcal{F}_Y(X)$  is isomorphic as

a  $\mathbb{F}$ -vector space to the tensor product  $Y \otimes_{\Delta} X$ . Actually,  $\mathcal{F}_Y(X)$  is isomorphic as a  $\mathbb{F}$ -algebra to  $Y \otimes_{\Delta} X$  under the product

$$(y \otimes x)(z \otimes w) = y\langle x, z \rangle \otimes w = y \otimes \langle x, z \rangle w.$$

**Definition 1.4.** By a *finitary associative  $\mathbb{F}$ -algebra* we mean any subalgebra  $A$  of  $\mathcal{F}(X)$  for some vector space  $X$  over  $\mathbb{F}$ .

**Lemma 1.5.** *If  $A$  is finitary, then  $aAa$  is finite dimensional for any  $a \in A$ .*

*Proof.* Let  $A \leq \mathcal{F}(X)$ , with  $X$  being a vector space over  $\mathbb{F}$ . It follows from (1.2) and (1.3) that for any  $a \in A$ ,  $aAa \subset a\mathcal{F}(X)a \cong a(X^* \otimes_{\mathbb{F}} X)a = a^{\#}X^* \otimes_{\mathbb{F}} Xa$  is finite dimensional, with  $\dim_{\mathbb{F}} aAa \leq \text{rank}(a)^2$ .  $\square$

**Theorem 1.6.** *Let  $A$  be an associative  $\mathbb{F}$ -algebra.*

- (1) *If  $A$  is semiprime, then  $A$  is finitary if and only if for any element  $a \in A$  the subspace  $aAa$  is finite dimensional.*
- (2)  *$A$  is semiprime and finitary if and only if it is isomorphic to a direct sum of simple finitary  $\mathbb{F}$ -algebras.*
- (e)  *$A$  is simple and finitary if and only if it is isomorphic to some  $\mathcal{F}_Y(X)$ , where  $(X, Y, \langle \cdot, \cdot \rangle)$  is a pair of dual vector spaces over a finite-dimensional division  $\mathbb{F}$ -algebra  $\Delta$ , equivalently,  $A$  is simple and contains an idempotent  $e$  such that  $eAe$  is a finite-dimensional division  $\mathbb{F}$ -algebra.*

*Proof.* By Lemma 1.5, if  $A$  is finitary then for any  $a \in A$ ,  $\dim_{\mathbb{F}} aAa < \infty$ .

Suppose now that  $A$  is semiprime and  $\dim_{\mathbb{F}} aAa < \infty$  for any  $a \in A$ . By [5, Theorem 2.3], the element  $a$  belongs to the socle of  $A$ , so, by socle theory for semiprime rings (see [7] or [4]),  $A = \bigoplus M_i$  is a direct sum of ideals each of which is a simple algebra with minimal one-sided ideals. Moreover, for any division idempotent  $e \in M_i$  the division algebra  $eM_i e = eAe$  is finite dimensional. Now it follows from the structure theorem for simple associative rings with minimal one sided ideals (see [4, Theorems 4.3.7 and 4.3.8]) that  $M_i$  is isomorphic to some  $\mathcal{F}_Y(X)$ , where  $(X, Y, \langle \cdot, \cdot \rangle)$  is a pair of dual vector spaces over a finite-dimensional division  $\mathbb{F}$ -algebra  $\Delta$ . For any  $a \in \mathcal{F}_Y(X)$ ,  $\dim_{\mathbb{F}} Xa = (\dim_{\Delta} Xa)(\dim_{\mathbb{F}} \Delta) < \infty$ , so  $\mathcal{F}_Y(X) \leq \mathcal{F}(X_{\mathbb{F}})$  is a simple finitary associative  $\mathbb{F}$ -algebra, equivalently,  $A$  is simple and contains an idempotent  $e$  such that  $eAe$  is a finite-dimensional division  $\mathbb{F}$ -algebra. Finally, it is easy to see that any direct sum of simple finitary associative  $\mathbb{F}$ -algebras is a semiprime finitary associative  $\mathbb{F}$ -algebra. This completes the theorem of the theorem.  $\square$

**Corollary 1.7.** *Let  $R$  be a simple associative ring. Then  $R$  is finitary, as an algebra over its centroid, if and only if  $R$  satisfies a generalized polynomial identity.*

*Proof.* Let  $\mathbb{F}$  denote the centroid of  $R$ , which is a field. By the theorem above,  $R$  is finitary as an algebra over  $\mathbb{F}$  if and only if it contains an idempotent  $e$  such that  $eRe$  is a finite dimensional division  $\mathbb{F}$ -algebra, equivalently, by [4, Theorem 6.1.6], the simple ring  $R$  satisfies a generalized polynomial identity.  $\square$

## 2. GENERAL FACTS ON QUADRATIC JORDAN ALGEBRAS

In what follows  $\Phi$  will denote a ring of scalars, that is, a commutative associative ring with 1. Following the exposition given in [12], we begin by reminding the reader of the definition of quadratic Jordan algebra.

**2.1. Definitions.** A *unital (quadratic) Jordan algebra*  $J = (J, U, 1)$  over an arbitrary ring of scalars  $\Phi$  consists of a  $\Phi$ -module  $J$ , a distinguished element  $1 \in J$ , and a quadratic map  $U : J \rightarrow \text{End}_{\Phi}(J)$  such that if we denote the linearization of  $U$  by

$$V_{x,y}(z) = \{x, y, z\} = U_{x,z}(y) \quad (U_{x,z} = U_{x+z} - U_x - U_z)$$

then

$$(QJ1) \quad U_1 = \text{Id}$$

$$(QJ2) \quad U_x V_{y,x} = V_{x,y} U_x = U_{U(x)y,x}$$

$$(QJ3) \quad U_{U(x)y} = U_x U_y U_x$$

hold in all (free) scalar extensions; equivalently, all linearizations of these identities hold in  $J$  itself.

A *(quadratic) Jordan algebra* is just a  $\Phi$ -submodule  $J = (J, U; 2)$  of some unital Jordan algebra closed under the products  $U_x y$  and the square

$$x^2 = U_x 1,$$

in which case  $J$  imbeds in the *unital hull*.

$$\hat{J} = \Phi 1 \oplus J :$$

$$U_{\alpha 1 \oplus x}(\beta 1 \oplus y) = \alpha^2 \beta \oplus (\alpha^2 y + 2\alpha \beta x + \alpha x \circ y + \beta x^2 + U_x y),$$

where we denote the linearization of the square by

$$V_x(y) = x \circ y = U_{x,y} 1 \quad (= (x + y)^2 - x^2 - y^2.)$$

If  $\frac{1}{2} \in \Phi$  we can characterize these algebras axiomatically as the *linear Jordan algebras* with product  $x \cdot y$  satisfying

$$x \cdot y = y \cdot x, \quad (x^2 \cdot y) \cdot x = x^2 \cdot (x \cdot y)$$

Every quadratic Jordan algebra  $J$  gives rise to a *Jordan pair*  $V = (J, J)$  where  $Q_x^\sigma y = U_x^\sigma y$  for  $\sigma = \pm$  (see [8]). Hence every notion defined by Jordan pairs makes sense for quadratic Jordan algebras.

In what follows, by a Jordan algebra we will mean a quadratic Jordan algebra.

**2.2. Examples.** Most Jordan algebras come from associative algebras. Any associative  $\Phi$ -algebra  $A$ , with product denoted by juxtaposition, yields a Jordan algebra  $A^{(+)}$  via

$$U_x y = xyx, \quad x^2 = xx, \quad \{x, y, z\} = xyz + zyx, \quad x \circ y = xy + yx$$

(which is unital if  $A$  is). A Jordan algebra  $J$  is *special* if it is isomorphic to a subalgebra of some  $A^{(+)}$ ; otherwise,  $J$  is called *exceptional*. An important example of special Jordan algebras is a *hermitian algebra*

$$H(A, *) = \{x \in A : x^* = x\} \leq A^{(+)}$$

of self-adjoint elements in an associative algebra  $A$  with involution  $*$ . More generally, we must consider *ample hermitian algebras*

$$H_0 = H_0(A, *) \leq H(A, *)$$

such that  $aH_0a^* \subset H_0$  for all  $a \in A$  and all traces  $a + a^*$  and norms  $aa^*$  lie in  $H_0$ . If  $\frac{1}{2} \in \Phi$  then the only ample subspace is the whole  $H(A, *)$ .

Another important example is a (unital Jordan) *Clifford algebra*, which lives in the associative clifford algebra  $C(Q, X, 1)$  of a quadratic form  $Q$  with base point 1 on a vector space  $X$  over a field  $\mathbb{F}$ :

$$J = J(Q, X, 1) \leq C(Q, X, 1)^{(+)} : \quad U_x y = Q(x, \bar{y})x - Q(x)\bar{y},$$

$\bar{y} = Q(y, 1)1 - y$  for  $x, y \in X$ ).

More generally, we can consider an *outer ideal*  $I$  of  $J(Q, X, 1)$  containing 1,  $U(J)I + \{J, J, I\} \subset I$ . If the characteristic of  $\mathbb{F}$  is not 2, then an outer ideal is an ideal.

A Jordan algebra is *i-special* if it satisfies all the identities of special Jordan algebras (equivalently, is a homomorphic image of a special Jordan algebra). A Jordan algebra is *i-exceptional* if it is not i-special. The basic i-exceptional Jordan algebras are the 27-dimensional *Albert algebras* (see [10]).

**2.3. Alternating algebras.** Let  $(X, Y, \langle \cdot, \cdot \rangle)$  be a pair of dual vector spaces over a division  $\Phi$ -algebra  $\Delta$ . If the  $\Phi$ -algebra  $\mathcal{F}_Y(X)$  has an involution  $*$ , then  $\Delta$  has an involution  $-$ ,  $X$  has either a nondegenerate hermitian (and nonalternating) inner product or a nondegenerate alternating inner product, denoted in both cases by  $\langle \cdot, \cdot \rangle$ , with the involution  $*$  being the adjoint  $\#$  with respect to the inner product. Notice that  $(X, \langle \cdot, \cdot \rangle)$  yields a pair of dual vector spaces  $(X, X, \langle \cdot, \cdot \rangle)$  where the second  $X$  is regarded as right vector space over  $\Delta$  by defining  $x \cdot \alpha = \bar{\alpha}x$  ( $x \in X, \alpha \in \Delta$ ).

Suppose that  $(X, \langle \cdot, \cdot \rangle)$  is alternating over a field  $\mathbb{F}$ . Following [11, p. 460], a linear mapping  $a \in \mathcal{F}_X(X)$  is called an *alternating mapping* if  $\langle x, xa \rangle = 0$  for each  $x \in X$ . We write  $Alt(X, \langle \cdot, \cdot \rangle)$  to denote the set of all alternating mappings. As observed in [6], if the characteristic of  $\mathbb{F}$  is not 2, then  $Alt(X, \langle \cdot, \cdot \rangle) = H(\mathcal{F}_X(X), *)$ .

**2.4. Homotopes, isotopes and local algebras.** Definitions and notation of this paragraph are taking from [6, 8].

For any element  $b$  in a Jordan algebra  $J$  we obtain a new Jordan algebra structure  $J^{(b)}$  in the  $\Phi$ -submodule  $J$  called the *b-homotope* of  $J$  by taking the operations

$$U_x^{(b)} = U_x U_b, \quad x^{(2,b)} = U_x b.$$

If  $J$  is unital and  $b$  is invertible, then  $J^{(b)}$  is a unital Jordan algebra called the *b-isotope* of  $J$ , with  $b^{-1}$  as its unit element.

*Isotopes of nonunital Jordan algebras*  $J$  are defined via a unital Jordan algebra  $K$  containing  $J$  as a subalgebra and an invertible element  $b \in K$  satisfying

$$U_b J = J, \quad U_J b \subset J.$$

For instance, the above conditions are satisfied if  $J$  is an ideal of  $K$ ; in particular, if  $K$  is the unital hull of  $J$ .

Let  $b$  be an arbitrary element of  $J$ . Then the  $\Phi$ -submodule  $\text{Ker}_J(b)$  of  $J$  defined by

$$\text{Ker}_J(b) = \{z \in J : U_b z = U_b U_z b = 0\}$$

is an ideal of the Jordan algebra  $J^{(b)}$  and the quotient algebra  $J/\text{Ker}_J(b)$  is called the *local algebra of  $J$  at  $b$*  and denoted by  $J_b$ . If  $\frac{1}{2} \in \Phi$ , then condition  $U_b U_z b = 0$  is superfluous.

**2.5. Inner ideals and socle.** An *inner ideal* of a Jordan algebra  $J$  is a  $\Phi$ -submodule  $B$  of  $J$  such that  $U_b J \subset B$  for any  $b \in J$ . Any element  $x \in J$  yields the *principal inner ideal determined by  $x$*   $U_x J$  and the *inner ideal generated by  $x$*   $\Phi x + U_x J$ . These inner

ideals coincide if and only if  $x$  is a von Neumann regular element of  $J$ . The *socle*  $\text{Soc}(J)$  of a Jordan algebra  $J$  is defined as the sum of its minimal inner ideals.

A Jordan algebra  $J$  is *nondegenerate* if  $U_x J = 0$  implies  $x = 0$ . Simple Jordan algebras are nondegenerate (see [1]). If  $J$  is nondegenerate, then  $\text{Soc}(J)$  is a direct sum of ideals each of which is a simple Jordan algebra coinciding with its socle (see [9] or [15]) Moreover, an element  $x \in J$  lies in the socle if and only if  $\Phi x + U_x J$  has descending chain condition on principal inner ideals [9, Theorem 1].

Simple Jordan algebras with minimal inner ideals were classified in [6] as a refinement of the more general structure theorem of K. McCrimmon and E. Zelmanov for simple Jordan algebras (see [12]).

**Theorem 2.6.** [6, Theorem 2] *Every simple Jordan algebra containing minimal inner ideals is up to isotopy one of the following:*

- (i) *a Jordan algebra of Clifford or Albert type,*
- (ii) *a Jordan algebra  $\mathcal{F}_Y(X)^{(\cdot)}$ , where  $(X, Y, \langle \cdot, \cdot \rangle)$  is a pair of dual vector spaces over a division  $\Phi$ -algebra  $\Delta$ ,*
- (iii) *an ample hermitian algebra  $H_0(\mathcal{F}_X(X), *)$  relative to a nondegenerate hermitian (and nonalternating) inner product space  $(X, \langle \cdot, \cdot \rangle)$  over a division  $\Phi$ -algebra  $\Delta$  with involution, and where  $*$  is the adjoint involution, or*
- (iv)  *$\text{Alt}(X, \langle \cdot, \cdot \rangle)$  for a nondegenerate alternating inner product space  $(X, \langle \cdot, \cdot \rangle)$  over a field  $\mathbb{F}$  which is a  $\Phi$ -algebra.*

### 3. FINITARY JORDAN ALGEBRAS

By analogy with the associative and Lie cases, we could define a finitary (quadratic) Jordan  $\mathbb{F}$ -algebra  $J$  as a subalgebra of the Jordan algebra  $\mathcal{F}(X)^{(\cdot)}$  for some vector space  $X$  over  $\mathbb{F}$ . But such a definition would exclude the exceptional Jordan algebras.

**Definition 3.1.** A Jordan  $\mathbb{F}$ -algebra  $J$  is *finitary* if the principal inner ideal  $U_a J$  is finite dimensional for each element  $a \in J$ .

**Lemma 3.2.** *If  $J \leq \mathcal{F}(X)^{(\cdot)}$  for some  $\mathbb{F}$ -vector space  $X$ , then  $J$  is finitary.*

*Proof.* The same proof of Lemma 1.5 works. For any  $a \in J$ ,  $U_a J \subset U_a \mathcal{F}(X) = a \mathcal{F}(X) a$ . □

**Remarks 3.3.** Notice that the converse of Lemma 3.2 does not hold. Consider a 27-dimensional Albert algebra. Notice also that, according to our definition, a unital

Jordan algebra is finitary if and only if it is finite dimensional; in particular, a Clifford algebra is finitary if and only if it is finite dimensional.

Our aim is to classify nondegenerate finitary (quadratic) Jordan algebras  $J$  over an arbitrary field  $\mathbb{F}$ . The theorem below reduces the study to the case that  $J$  is simple.

**Theorem 3.4.** *Let  $J$  be a Jordan  $\mathbb{F}$ -algebra. Then  $J$  is finitary and nondegenerate if and only if it is a direct sum of simple finitary Jordan algebras.*

*Proof.* Let  $J = \bigoplus J_i$  be a direct sum of ideals each of which is a simple finitary Jordan  $\mathbb{F}$ -algebras. But simple Jordan algebras are nondegenerate [1] and it is easy to check that direct sums preserve nondegeneracy, so  $J$  is nondegenerate. Now let  $a \in J$ . Then  $a = a_{i_1} + \cdots + a_{i_r}$  where each summand  $a_{i_k}$  belongs to some  $J_{i_k}$ . Since the ideals  $J_i$  are mutually orthogonal, we have that  $U_a J = U(a_{i_1})J_{i_1} \oplus \cdots \oplus U(a_{i_r})J_{i_r}$  is finite dimensional, which proves that  $J$  is finitary.

Suppose conversely that  $J$  is nondegenerate with  $U_a J$  being finite dimensional for any element  $a \in J$ . It follows from Loos' elemental characterization of the socle [9, Theorem 1] that  $J$  coincides with its socle and therefore, by [9, Theorem 2],  $J$  is a direct sum of ideals each of which is a simple Jordan algebra, clearly finitary. This completes the proof of the theorem.  $\square$

**Theorem 3.5.** *Every simple finitary Jordan  $\mathbb{F}$ -algebra is up to isotopy one of the following:*

- (i) *a 27-dimensional Albert algebra over a finite field extension  $\mathbb{K}$  of  $\mathbb{F}$ ,*
- (ii) *a finite-dimensional Clifford Jordan algebra  $J$ , defined by a nondegenerate quadratic form on a vector space over a finite field extension  $\mathbb{K}$  of  $\mathbb{F}$ , or an outer ideal  $I$  of  $J$  containing 1.*
- (iii) *an ample hermitian algebra  $H_0(\mathcal{F}_X(X), *)$  relative to a nondegenerate hermitian (and nonalternating) inner product space  $(X, \langle \cdot, \cdot \rangle)$  over a finite-dimensional division  $\mathbb{F}$ -algebra  $\Delta$  with involution, and where  $*$  is the adjoint involution, or*
- (iv)  *$\text{Alt}(X, \langle \cdot, \cdot \rangle)$  for a nondegenerate alternating inner product space  $(X, \langle \cdot, \cdot \rangle)$  over a finite field extension  $\mathbb{K}$  of  $\mathbb{F}$ .*

*Proof.* Each of the Jordan algebras listed above is simple and finitary. Suppose then that  $J$  is a simple finitary Jordan  $\mathbb{F}$ -algebra. As proved in Theorem 3.4,  $J$  coincides with its socle. Thus we only need to look at the list of simple Jordan algebras given in Theorem 2.6 and identify the finitary ones.  $\square$

**Remark 3.6.** Finitary Jordan algebras are closely related to Jordan algebras with PI-elements, as defined and studied by F. Montaner in [13, 14]. In terms of local algebras (see 2.4), our definition of a finitary Jordan algebra  $J$  can be rephrased by saying that the local algebra  $J_a$  is finite dimensional for any element  $a \in J$ , while, according to [13], an element  $a \in J$  is PI if the local algebra  $J_a$  satisfies a polynomial identity. Notice that by [13, Proposition 2.6], simple unital Jordan algebras with a nonzero PI-element have finite capacity, and in the particular case of a Jordan algebra  $J = A^{(+)}$ , where  $A$  is a central simple associative algebra,  $J$  is finitary if and only if it contains a nonzero PI-element.

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