

AMS/IP

**Studies in
Advanced
Mathematics**

S.-T. Yau, Series Editor

**Complex Differential
Geometry**

Fangyang Zheng

American Mathematical Society · International Press

AMS/IP

<https://doi.org/10.1090/amsip/018>

Studies in Advanced Mathematics

Volume 18

Complex Differential Geometry

Fangyang Zheng

American Mathematical Society • International Press



Shing-Tung Yau, Managing Editor

2000 *Mathematics Subject Classification*. Primary 53 01;
Secondary 53-00, 53-02, 53C55.

The author was supported in part by NSF Grant # DMS-9703884,
NSA Grant MDA904-98-1-0036, and a fellowship from the Alfred P. Sloan Foundation.

Library of Congress Cataloging-in-Publication Data

Zheng, Fangyang, 1962–

Complex differential geometry / Fangyang Zheng.

p. cm. — (AMS/IP studies in advanced mathematics, ISSN 1089-3288 ; v. 18)

Includes bibliographical references and index.

ISBN 0-8218-2163-6

1. Complex manifolds. 2. Geometry, Differential. I. Title. II. Series.

QA331.7.Z48 2000

516.3'6—dc21

00-056549

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Assistant to the Publisher, American Mathematical Society, P. O. Box 6248, Providence, Rhode Island 02940-6248. Requests can also be made by e-mail to reprint-permission@ams.org.

© 2000 by the American Mathematical Society and International Press. All rights reserved.

Reprinted with corrections by the American Mathematical Society, 2001

The American Mathematical Society and International Press retain all rights
except those granted to the United States Government.

Printed in the United States of America.

⊗ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.

Visit the AMS home page at URL: <http://www.ams.org/>

Visit the International Press home page at URL: <http://www.intlpress.com/>

10 9 8 7 6 5 4 3 2 1 05 04 03 02 01 00

To Qun and Alex

This page intentionally left blank

Contents

Preface	vii
Part 1. Riemannian Geometry	1
Chapter 1. Differentiable Manifolds and Vector Bundles	3
1.1. Differentiable Manifolds	3
1.2. Tangent Spaces and Vector Fields	5
1.3. Vector Bundles	8
1.4. Tangent Bundles and Tensor Fields	11
1.5. The Topology of Smooth Manifolds	14
1.6. Lie Groups and Lie Algebras	16
Appendix: Topology, Homotopy and Covering Spaces	20
Exercises	23
Chapter 2. Metric, Connection, and Curvature	29
2.1. Metric, Connection, and Curvature	29
2.2. Linear Connections and Geodesics	31
2.3. Riemannian Metrics and Riemannian Connections	35
2.4. Sectional, Ricci and Scalar Curvatures	37
2.5. Cartan's Structure Equations and Examples	40
Exercises	44
Chapter 3. The Geometry of Complete Riemannian Manifolds	49
3.1. Riemannian Distance	49
3.2. Completeness and Hopf-Rinow Theorem	53
3.3. Jacobi Fields and Conjugate Points	56
3.4. Cartan-Ambrose-Hicks Theorem and Space Forms	64
3.5. Homogeneous and Symmetric Spaces	66
3.6. Hodge Theorem and Comparison Theorems	70
Exercises	74
Part 2. Complex Manifolds	81
Chapter 4. Complex manifolds and Analytic Varieties	83
4.1. Holomorphic Functions of One or More Complex Variables	83
4.2. Definition and Examples of Complex Manifolds	86
4.3. The Almost Complex Structure	89
4.4. More Examples	92
4.5. Hypersurfaces and Analytic Subvarieties	95
4.6. Divisors and Analytic Cycles	99

Exercises	101
Chapter 5. Holomorphic Vector Bundles, Sheaves and Cohomology	105
5.1. Holomorphic Vector Bundles	105
5.2. Sheaves	108
5.3. Sheaf Cohomology Groups	112
5.4. Holomorphic Line Bundles	115
5.5. Chern Classes	119
Exercises	123
Chapter 6. Compact Complex Surfaces	127
6.1. The Topological Invariants	127
6.2. The Kodaira Dimension and the Algebraic Dimension	132
6.3. Examples of Surfaces	137
6.4. Enriques-Kodaira Classification Theory for Surfaces	142
Exercises	151
Part 3. Kähler Geometry	155
Chapter 7. Hermitian and Kähler Metrics	157
7.1. Connections on Vector Bundles and Their Curvature	157
7.2. Chern Forms of a Complex Vector Bundle	160
7.3. Hermitian Bundles	166
7.4. Hermitian and Kähler Metrics on Complex Manifolds	170
7.5. The Curvature of a Hermitian or Kähler Metric	176
7.6. Wu's Theorem, Schwarz Lemma and Hartogs Phenomenon	181
Exercises	187
Chapter 8. Compact Kähler Manifolds	191
8.1. Hodge Theorem and Hodge Decomposition	191
8.2. The Hard Lefschetz Theorem	195
8.3. Kodaira Vanishing and Embedding Theorems	200
8.4. Ample Subvarieties and Ample Vector Bundles	204
8.5. Hermitian Symmetric Spaces and Kähler C-Spaces	208
8.6. The Hartshorne-Frankel Conjecture	213
Exercises	219
Chapter 9. Kähler Geometry	221
9.1. Calabi's Conjecture and Kähler-Einstein Metrics	221
9.2. Corollaries of Yau's Theorems	225
9.3. Invariant Metrics	230
9.4. Harmonic Maps and the Rigidity Theorems	236
9.5. Non-positively Curved Kähler Surfaces	242
Exercises	249
Bibliography	255
Index	259

Preface

Complex manifold has been sitting in the overlap of quite a few branches of mathematics, such as differential geometry, algebraic geometry, several complex variables, global analysis, topology, algebraic number theory, mathematical physics, etc..

On one hand, complex manifolds provide a rich class of geometric objects. For instance, the (common) zero locus of any (generic set of) complex polynomials is always a complex manifold. In fact, until very recently, the connected sum of simply connected algebraic surfaces has been the only known examples of simply connected smooth 4-manifolds. On the other hand, complex manifolds behave rather differently than generic smooth manifolds. They are much more coherent. The existence of a complex structure often imposes topological consequences on the underlying smooth manifold. For instance, only a rather special kind of compact smooth manifold can admit a Kählerian complex structure. As another example, the group of biholomorphisms of a compact complex manifold is finite dimensional, while the diffeomorphism group of a smooth manifold is always infinite dimensional.

This rich yet restrictive character makes complex manifold a rather special and interesting object of study. This book is about the differential geometric aspects of complex manifolds. Our main goal is to provide a self-contained, comprehensive approach to the subject, and saving the readers from traveling back and forth between two sets of books in learning the subject. Also, in the last chapter, we include some more recent results in the field, most of which are not yet collected in textbooks.

The book is divided into three parts, each contains three chapters. The first part contains standard materials from general topology, differentiable manifold, and basic Riemannian geometry. The second part discuss complex manifolds and analytic varieties, sheaves and holomorphic vector bundles, as well as a brief account of the surface classification theory, which provide the readers with some concrete examples of complex manifolds. The last part is the main purpose of the book, in which we discuss metric, connection, curvature and the various roles they played in the study of complex manifolds. We also collected some exercises at the end of each chapter. Most of them are ranging from straight forward to medium challenging, and serve the purpose of enhancing the comprehension of the topics. A few of them are more challenging, and could be regarded as questions leading to supplementary readings.

Part of the materials in this book have been used by the author in the lectures of a graduate course given at Ohio State University in 1998/99, as well as in a mini course taught in the 1998 summer school at Nanjing University. We would like to thank the students who participated for their feedbacks and suggestions, which made the book relatively more focused and readable. We would also like to thank Eric Loubeau, G. Tian, H. Wu and S-T Yau for several valuable corrections and suggestions towards the manuscript. Finally, we would like to thank NSF, NSA, Sloan Foundation and the Ohio State University for the financial support, which made the writing of this book possible.

Fangyang Zheng

Bibliography

[BOOKS]

- [B] A. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin, New York, 1987.
- [BC] F. Brickell and R.S. Clark, *Differentiable Manifolds*, Van Nostrand Reinhold Co., London, 1979.
- [BPV] W. Barth, C. Peters and A. Van de Ven, *Compact Complex Surfaces*, Springer-Verlag, Berlin, New York, 1984.
- [C] C. Chevalley, *Anneaux de Chow et Applications*, Séminaire Chevalley, Secrétariat Math., Paris (1958).
- [C1] C. Chevalley, *Theory of Lie groups*, Princeton University Press, 1946.
- [CE] J. Cheeger and D. Ebin, *Comparison Theorems in Riemannian Geometry*, North-Holland, Amsterdam, 1975.
- [G] S. Goldberg, *Curvature and homology*, Academic Press, New York and London, 1962.
- [GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, New York, 1978.
- [GR] R. Gunning and Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall Inc., Englewood Cliffs, NJ, 1965.
- [GW] R. Greene and H. Wu, *Function Theory on Manifolds which Possess a Pole*, Lecture Notes in Mathematics, Volume 699, Springer-Verlag, 1979.
- [H] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977.
- [He] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York, 1978.
- [K] S. Krantz, *Function Theory of Several Complex Variables*, John Wiley & Sons, New York, 1982.
- [KN] S. Kobayashi and N. Nomizu, *Foundations of Differential Geometry I, II*, Interscience, New York, 1963, 1969.
- [M] N. Mok, *Metric Rigidity Theorems on Hermitian Locally Symmetric Manifolds*, Series in Pure Math., Vol. 6, World Scientific, Singapore-New Jersey-London-Hong Kong, 1989.
- [Mi] J. Milnor, *Morse Theory*, Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51 Princeton University Press, 1963.
- [MS] J. Milnor and J. Stasheff, *Characteristic Classes*, Annals of Mathematics Studies, No. 76. Princeton University Press, 1974.
- [N] R. Narasimhan, *Analysis on Real and Complex Manifolds*, Advanced Studies in Pure Mathematics, Vol. 1; North-Holland, Amsterdam, 1968.
- [SY] R. Schoen and S-T Yau, *Differential Geometry*, Ke Xue Press, Beijing, 1998 (in Chinese).
- [WC] H. Wu and W. Chen, *Topics in Riemannian Geometry*, Peking University Press, Beijing, 1993 (in Chinese).
- [WSY] H. Wu, C-L Shen and Y-L Yu, *Riemannian Geometry*, Peking University Press, Beijing, 1989 (in Chinese).

[ARTICLES]

- [Al] S.I. Al'ber, *Space of mappings into a manifold with negative curvature*, Sov. Math. Dokl. **9** (1967), 6–9.
- [Ar] D. Arapura, *Threefolds with semipositive tangent bundle*, Ph.D. Thesis, Columbia University, 1985.
- [Au] T. Aubin, *Equations du type de Monge-Ampère sur les variété Kähle-riennes compactes*, C.R. Acad. Sci. Paris **283** (1976), 119–121.
- [AS] W. Ambrose and I.M. Singer, *A theorem on holonomy*, Trans. Amer. Math. Soc., **243** (1953), 428–443.
- [AV] A. Andreotti and E. Vesentini, *Carleman's estimates for the Laplace-Beltrami operator on complex manifolds*, Publ. Math. I.H.E.S. **25** (1965), 81–130.
- [Ba] S. Bando, *On three-dimensional compact Kähler manifolds of nonnegative bisectional curvature*, J. Diff. Geom. **19** (1984), 283–297.
- [Be] M. Berger, *Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes*, Bull. Soc. Math France, **83** (1955), 279–330.
- [Bo] F.A. Bogomolov, *Classification of surfaces of class VII_0 with $b_2 = 0$* , Math. USSR-Izv., **10** (1976), 255–269.
- [Bo1] F.A. Bogomolov, *Surfaces of class VII_0 and affine geometry*, Math. USSR-Izv., **21** (1983), 31–73.
- [BK] E. Bedford and M. Kalka, *Foliations and the complex Monge-Ampère equation*, Comm. Pure Appl. Math. **30** (1977), 543–572.
- [BM] E. Bierstone and P. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), 207–302.
- [Ca] E. Calabi, *The space of Kähler metrics*, Proc. Interat. Congress Math., Amsterdam, **2** (1954), 206–207.
- [Co] K. Corlette, *Archimedean superrigidity and hyperbolic geometry*, Ann. Math. **135** (1992), 165–182.
- [CC] H.D. Cao and B. Chow, *Compact Kähler manifolds with nonnegative curvature operator*, Invent. math. **83** (1986), 553–556.
- [CG] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Diff. Geom. **6** (1971), 119–128.
- [CP] F. Campana and Th. Peternell, *Projective manifolds whose tangent bundles are numerically effective*, Math. Ann. **289** (1991), 169–187.
- [CT] J. Carlson and D. Toledo, *Harmonic mappings of Kähler manifolds to locally symmetric spaces*, Publ. Math. I.H.E.S. **69** (1989), 173–201.
- [CY] S-Y. Cheng, S-T. Yau, *On the existence of a complete Kähler metric on non-compact complex manifolds and the regularity of Fefferman's equation*, Comm. Pure Appl. Math. **33** (1980), 507–544.
- [DGMS] P. Deligne, P. Griffiths, J. Morgan and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), 245–274.
- [DPS] J.P. Demailly, Th. Peternell and M. Schneider, *Compact complex manifolds with numerically effective tangent bundles*, J. Algebraic Geom. **3** (1994), 295–345.
- [E] P. Eberlein, *Rigidity of lattices of nonpositive curvature*, Erg. Th. Dynam. Sys., **3** (1983), 47–85.
- [F] T. Frankel, *Manifolds with positive curvatures*, Pacific J. Math. **11** (1961) 165–174.
- [FL] W. Fulton and R. Lazarsfeld, *Positive polynomials for ample vector bundles*, Ann. of Math **118** (1983), 35–60.
- [GrW] D. Gromoll and J. A. Wolf, *Some relations between the metric structure and the algebraic structure of the fundamental group in manifolds of nonpositive curvature*, Bull. A.M.S., **77** (1971), 545–552.
- [GS] M. Gromov and R. Schoen, *Harmonic maps into singular spaces and p -adic superrigidity for lattices in groups of rank one*, I.H.E.S. Publ. Math., **76** (1992), 165–246.
- [H1] R. Hartshorne, *Ample vector bundles*, Publ. Math. I.H.E.S. **29** (1966), 319–394.
- [Ham] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. **17** (1982), 255–306.
- [Hat] P. Hartman, *On homotopic harmonic maps*, Canadian J. Math. **19** (1967), 673–687.
- [Hi] Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. Math. **79** (1964), I: 109–203; II: 205–326.

- [Ho] L. Hörmander, L^2 -estimates and the existence theorems for the $\bar{\partial}$ -operator, *Acta Math.* **114** (1967), 89–152.
- [HSW] A. Howard, B. Smyth and H. Wu, *On compact Kähler manifolds of nonnegative bisectional curvature*, *Acta Math.* **147** (1981), 51–56.
- [I] S. Iitaka, *On algebraic varieties whose universal covering manifolds are complex affine 3-spaces*, *Bull. Amer. Math. Soc.* **78** (1972), 737–740.
- [JY] J. Jost and S-T Yau, *Harmonic mappings and Kähler manifolds*, *Math. Ann.* **262** (1983), 145–166.
- [JY1] J. Jost and S-T Yau, *A strong rigidity theorem for a certain class of compact complex analytic surfaces*, *Math. Ann.* **271** (1985), 143–152.
- [JY2] J. Jost and S-T Yau, *The strong rigidity of locally symmetric complex manifolds of rank one and finite volume*, *Math. Ann.* **275** (1986), 291–304.
- [JY3] J. Jost and S-T Yau, *On the rigidity of certain discrete groups and algebraic varieties*, *Math. Ann.* **278** (1987), 481–496.
- [JY4] J. Jost and S-T Yau, *Harmonic maps and Kähler geometry*, *Prospects in complex geometry* (Katata and Kyoto, 1989), 340–370, *Lecture Notes in Math.*, 1468, Springer, Berlin, 1991.
- [JY5] J. Jost and S-T Yau, *Harmonic maps and group representations*, *Differential geometry*, 241–259, *Pitman Monographs Surveys Pure Appl. Math.*, 52, Longman Sci. Tech., Harlow, 1991.
- [JY6] J. Jost and S-T Yau, *A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry*, *Acta Math.* **170** (1993), 221–254.
- [JY7] J. Jost and S-T Yau, *Harmonic mappings and algebraic varieties over function fields*, *Amer. J. Math.* **115** (1993), 1197–1227.
- [JY8] J. Jost and S-T Yau, *Applications of quasilinear PDE to algebraic geometry and arithmetic lattices*, *Algebraic geometry and related topics* (Inchon, 1992), 169–193, *Conf. Proc. Lecture Notes Algebraic Geom.*, I, Internat. Press, Cambridge, MA, 1993.
- [JY9] J. Jost and S-T Yau, *Harmonic maps and superrigidity*, *Tsing Hua lectures on geometry & analysis* (Hsinchu, 1990–1991), 213–246, *Internat. Press, Cambridge, MA*, 1997.
- [JY10] J. Jost and S-T Yau, *Harmonic maps and rigidity theorems for spaces of nonpositive curvature*, *Comm. Anal. Geom.*, **7** (1999), 681–694.
- [L] Y.C. Lu, *Holomorphic mappings of complex manifolds*, *J. Diff. Geom.* **2** (1968), 299–312.
- [LY] S. Lu and S.T. Yau, *Holomorphic curves in surfaces of general type*, *Proc. Natl. Acad. Sci. USA* **87** (1990), 80–82.
- [LYZ] J. Li, S.T. Yau and F. Zheng, *A simple proof of Bogomolov’s theorem on class VII_0 surfaces with $b_2 = 0$* , *Illinois J. Math.*, **34** (1990), 217–220.
- [LYZ1] J. Li, S.T. Yau and F. Zheng, *On projectively flat Hermitian manifolds*, *Comm. Anal. Geom.* **2** (1993), 103–110.
- [Ma] Y. Matsushima, *Sur la structure du groupe d’homéomorphismes analytiques d’une certaine variété kählerienne*, *Nagoya Math. J.* **11** (1957), 145–150.
- [M1] N. Mok, *Compact Kähler manifolds of nonnegative holomorphic bisectional curvature*, *Complex analysis and algebraic geometry* (Gottingen, 1985), 90–103,
- [M2] N. Mok, *The holomorphic and antiholomorphic character of harmonic maps into irreducible compact quotients of polydiscs*, *Math Ann.* **272** (1985), 197–216.
- [M3] N. Mok, *Strong rigidity of irreducible quotients of polydiscs of finite volume*, *Math Ann.* **282** (1988), 555–578.
- [M4] N. Mok, *The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature*, *J. Differential Geom.* **27** (1988), 179–214.
- [Mo] S. Mori, *Projective manifolds with ample tangent bundles*, *Ann. Math.* **110** (1979), 593–606.
- [MSY] N. Mok, S-T Siu and S-K Yeung, *Geometric superrigidity*, *Invent. math.* **113** (1993), 57–83.
- [Na] I. Nakamura, *On surfaces of class VII_0 with curves. II*, *Tōhoku Math. J.* **42** (1990), 475–516.
- [NN] A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, *Ann. Math.* **65** (1957), 391–404.
- [R] H. Royden, *Remarks on the Kobayashi metric*, *Several Complex Variables II* (Maryland 1970), *Lecture Notes in Mathematics*, Volume 185, Springer-Verlag, (1971), 125–137.
- [S] J. Sampson, *Applications of harmonic maps to Kähler geometry*, *Contemp. Math.* **49** (1986), 125–134.
- [Si] J. Simons, *On the transitivity of holonomy systems*, *Ann. Math.* **76** (1962), 213–234.

- [Su] Y-T Siu, *The complex analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds*, Ann. Math. **112** (1980), 73–111.
- [SiY] Y.T. Siu and S.T. Yau, *Compact Kähler manifolds of positive bisectional curvature*, Invent. Math. **59** (1980), 189–204.
- [T] G. Tian, *Kähler-Einstein metrics on algebraic manifolds*, Transcendental Methods in Algebraic Geometry (Cetraro, 1994), 143–185, Lecture Notes in Math., 1646, Springer, Berlin, 1996.
- [Ts] K. Tsukada, *Einstein Kähler submanifolds with codimension 2 in a complex space form*, Math. Ann. **274** (1986), 503–516.
- [TY] G. Tian and S-T Yau, *Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry*, Mathematical Aspects of String Theory (S-T Yau ed.), p. 574–628, World Scientific, 1987.
- [U] M. Umehara, *Einstein Kähler submanifolds of a complex linear or hyperbolic space*, Tohoku Math. J. **39** (1987), 385–389.
- [W1] H. Wu, *On compact Kähler manifolds of nonnegative bisectional curvature*, Acta Math. **147** (1981), 57–70.
- [W2] H. Wu, *Old and new invariant metrics on complex manifolds*, Math. Notes **38** (1993), 640–682.
- [Wa] H.C. Wang, *Closed manifolds with homogeneous complex structures*, Amer. J. Math. **76** (1954), 1–32.
- [Wo] B. Wong, *A class of compact complex manifolds with negative tangent bundles*, Complex analysis of several variables (Madison, Wis., 1982), 217–223, Proc. Sympos. Pure Math., 41, Amer. Math. Soc., Providence, R.I., 1984.
- [Y] S.T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228.
- [Y1] S.T. Yau, *A general Schwarz lemma for Kähler manifolds*, Amer. J. Math. **100** (1978), 197–203.
- [Y2] S-T Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. USA **74** (1977) 1798–1799.
- [Y3] S.T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure Appl. Math. **31** (1978), 339–441.
- [Y4] S.T. Yau, *A splitting theorem and an algebraic geometric characterization of locally Hermitian symmetric spaces*, Comm. Anal. Geom. **1** (1993), 473–486.
- [Y5] S.T. Yau, *Review on Kähler-Einstein metric in algebraic geometry*, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), 433–443, Israel Math. Conf. Proc., 9, Bar-Ilan Univ., Ramat Gan, 1996.
- [YZ] S-T Yau and F. Zheng, *Negatively $\frac{1}{4}$ -pinched Riemannian metric on a compact Kähler manifold*, Invent. Math. **103** (1991), 527–535
- [Z] F. Zheng, *Projective threefolds with semipositive tangent bundles and threefolds with universal cover \mathbb{C}^3* , Ph.D. Thesis, Harvard University, 1989.
- [Z1] F. Zheng, *First Pontrjagin form, rigidity and strong rigidity of non-positively curved Kähler surfaces*, Math. Zeit., **220** (1995), 159–169.

Index

- (-1) curve, 134
- $\bar{\partial}$ -Poincaré lemma, 114, 123
- $\partial\bar{\partial}$ -lemma, 221
- L^2 -estimate for $\bar{\partial}$, 249
- p -form, 12
- (p, q) -form, 106
- VII_0 surface, 139

- A
- adequately negative, 241
- Adjoint representation, 19
- adjoint representation, 19
- adjunction formula, 131
- Albanese map, 130
- Albanese variety, 130
- Al'ber-Hartman thm, 237
- algebraic dimension, 134
- almost complex structure, 90
 - integrable, 91
 - torsion of, 91
- alternative product, 9, 10
- ample divisor, 118
- ample vector bundle, 206
- analytic cycle, 100
- analytic subvariety, 96
- arithmetic genus, 131
- associated meromorphic map, 117
- associated projective bundle, 107
- Aubin-Yau thm, 223
- automorphism group, 102

- B
- ball, 21
- basic index lemma, 60
- Berger's thm, 209
- Bergman metric, 175, 231
- beti number, 14
- Bianchi identities, 32
- bidegree, 11
- bidisk, 101
- Bieberbach thm, 74
- biholomorphism, 86
- bi-invariant metric, 76
- bimeromorphic map, 132

- birational map, 132
- birationally ruled surface, 138
- bisectional curvature, 177
- Bloch-Gieseker thm, 206
- blowing up,
 - at a point, 93
 - along a submanifold, 108
- Bochner identity, 76, 227, 228, 250
- branched covering, 204
- branching locus, 204
- Brody's thm, 233
- broken geodesic, 65
- boundary map, 14
- bounded symmetric domain, 209
 - classical 209, 210
 - exceptional 209
- bundle homomorphism, 11
- bundle map, 11
- Busemann function, 79

- C
- Calabi's conjecture, 223
- Calabi-Yau manifolds, 229
- Calabi-Yau thm, 229
- canonical line bundle, 107
- canonical line bundle formula, 142
- Carathéodory distance, 251
- Carlson-Toledo thm, 238
- Cartan decomposition, 69
- Cartan's structure equations, 41, 158, 172
- Cartan's thm, 114
- Cartan-Ambrose-Hicks thm, 65
- Cartan-Hadamard thm, 59
- Castelnuovo's criterion, 144
- Cauchy integral formula, 83, 85
- Cayley plane, 208
- chain complex, 15
- Chern character, 120
- Chern class, 120
- Chern form, 161
- Chern number, 121
- Chern number inequality, 226, 243
- Chern root, 121
- Chern vector, 211, 247

- Chern-Hirzebruch cone, 247
 Christoffel symbol, 35
 Chow group, 100
 classifying space, 164
 closed subset, 20
 coboundary map, 14
 cochain complex, 15, 112
 coframe, 41
 coherent sheaf, 111
 compact (set, space), 21
 compact homogeneous Kähler space, 212
 compact surface, 5
 orientable, 5
 non-orientable, 5
 complete intersection, 123
 complete metric space, 53
 complete Riemannian manifold, 54
 complex analytic function, 83
 complex analytic space, 98
 complex curvature operator, 180
 complex Laplacian, 183
 complex general linear group, 17
 complex manifold, 86
 complex projective space, 87
 complex space form, 175, 178
 complex structure, 89
 complex submanifold, 88
 complex surface, 127
 complex torus, 92
 conformal structure, 92
 conjugate point, 58
 connected component, 21
 connected (set, space), 21
 connected-sum, 4, 5 26
 connection, 29, 157
 compatible with complex structure, 166
 compatible with a metric, 30, 157
 connection matrix, 30, 158
 connection on induced bundles, 159, 160
 constant sheaf, 109
 continuous map, 21
 convex neighborhood, 52
 convexity radius, 53
 cotangent bundle, 11
 countable base, 21
 covariant differentiation, 29
 covering map, 4
 covering space, 22
 cubic surface, 153
 cup product, 15
 curvature decomposition, 47, 225
 scalar part, 47, 225
 traceless Ricci part, 47, 225
 curvature (1, 1)-form, 176
 curvature matrix, 30, 158
 curvature of a Hermitian bundle, 176, 177
 curvature of a Hermitian subbundle, 176
 curvature operator, 37
 curvature tensor, 30
 cut locus, 63
 CW-complex, 24
 cyclic covering, 153
- D**
 de Rham cohomology, 12
 de Rham decomposition thm, 56
 de Rham factor, 56
 de Rham thm, 15, 115
 Demailly-Peternell-Schneider thm, 218
 differentiable manifold, 3
 differentiable map, 3
 differentiable sphere thm, 74
 differentiable structure, 3
 differential (of a map), 7
 differential form, 12
 direct image sheaf, 123
 distance (function, space), 23, 49
 distribution, 13
 divisor, 99
 divisor class group, 99
 Dolbeault cohomology group, 106, 114
 domain in \mathbb{C}^n , 87
 dual connection, 159
 dual frame, 159
- E**
 Eells-Sampson thm, 237
 effective divisor, 131
 Einstein metric, 45, 77
 elliptic curve, 87
 elliptic surface, 135
 energy density function, 236
 energy functional, 236
 Enriques surface, 140
 Enriques-Kodaira classification thm, 143
 Euclidean de Rham factor, 56
 Euclidean metric, 174
 Euclidean space, 4, 41
 Euclidean symmetric space, 70
 Euler characteristic, 122
 Euler class, 119
 Euler number, 14
 Euler sequence, 107
 exceptional divisor, 108, 136
 exponential map, 34
 of a Lie group, 19
 exponential sequence, 110
 exterior algebra, 12
 exterior differentiation, 12
 exterior product, 9, 10
- F**
 Fano manifold, 218
 Fano surface, 153
 Feder's thm, 126
 Fermat hypersurface, 188
 first Ricci, 181

- fine sheaf, 113
- finite branched covering, 204
- fixed components of a linear system, 131
- flag, 89
- foliation, 13
- frame, 9
- Frankel's conjecture, 214
- free action (by a group), 23
- Frobenius's thm, 13
- Fubini-Study metric, 174
- Fulton-Lazarsfeld thm, 208
- function field, 99
- fundamental class, 100
- fundamental cycle, 100
- fundamental group, 22
- G
- Gauss equation, 47
- Gauss-Bonnet thm, 75
- general linear group, 17
- general type surface, 137
- generalized Hartshorne conjecture, 218
- generic fiber, 130
- genus of a curve, 123
- geodesic, 33
- geodesic symmetry, 68
- geodesically complete, 54
- geometric genus (of a complex surface), 127
- Gieseker coarse moduli space, 150
- good neighborhood, 34
- Grassmannian, 88
- Grauert's criterion, 132, 205
- Green's operator, 192
- Griffiths-Shiffman thm, 186
- Grothendieck group, 122
- Grothendieck's thm, 214
- H
- Hamilton's heat flow, 216
- Hard Lefschetz thm, 196
- harmonic form, 73
- harmonic (p, q) -forms, 192
- harmonic map, 236
- Hartogs extension thm, 103
- Hartogs phenomenon, 186
- Hartshorne's conjecture, 214
- Hausdorff, 21
- Hermitian bundle, 166
- Hermitian metric, 157, 170
- Hermitian negative, 237
- Hermitian non-positive, 237
- Hermitian subbundle, 167
- Hermitian symmetric space, 209
- Hernandez-Yau-Zheng thm, 239
- Hirzebruch proportionality principle, 211
- Hirzebruch surface, 139
- Hirzebruch-Riemann-Roch thm, 122
- Hirzebruch-Riemann-Roch inequality, 127
- Hitchin's thm, 251
- Hodge conjecture, 195
- Hodge decomposition thm, 194
- Hodge decomposition on forms, 192
- Hodge number, 192
- Hodge star operator, 72, 191
- Hodge thm, 72, 192
 - the bundle version of, 193
- holomorphic coordinate, 86
- holomorphic cotangent bundle, 105
- holomorphic fibration, 130
- holomorphic function, 83
- holomorphic map, 85
- holomorphic modification, 132
- holomorphic p -form, 106, 198
- holomorphic sectional curvature, 177
- holomorphic tangent bundle, 105
- holomorphic vector bundle, 105
- holomorphically convex, 94
- holonomy group, 31
- holonomy thm, 45
- homeomorphism, 21
- homogeneous complex manifold, 212
- homogeneous coordinate, 87
- homogeneous Kähler space, 212
- homogeneous space, 66
- homology group, 14, 31
- homotopy, 22
- homotopy group, 25
- homotopy equivalence, 25
- Hopf manifold, 92, 93
- Hopf surface, 140
- Hopf-Rinow thm, 54
- Howard-Smyth-Wu thm, 215
- hyperbolic space, 43, 44
- hyperelliptic surface, 141
- hyperplane line bundle, 107
- hypersphere, 188
- hypersurface, 95
- Hurwitz formula, 123
- I
- ideal sheaf, 111
- identity component, 16
- Iitaka conjecture, 101
- Iitaka-Kodaira dimension, 132
- image sheaf, 110
- indeterminacy, 99
- index formula, 59
- induced connection, 159, 160
- injectivity radius, 50
- Inoue surface, 154
- integrable, 91
- integral curve, 6
- integral domain, 96
- intersection pairing, 16
- invariant metric, 230
- invariant polynomial, 161
- invertible sheaf, 111, 116

- irreducible component, 95, 97
- irreducible hypersurface, 95
- irreducible Riemannian manifold, 56
- irreducible subvariety, 97
- irreducible symmetric space, 70
- irregularity, 127
- isometry group, 66
- isothermal coordinate, 45
- isotropy subgroup, 66
- Iwasawa manifold, 199

- J
- Jacobi equation, 56
- Jacobi field, 56
- Jacobi identity, 18

- K
- K -3 surface, 140
- Kähler C -space, 212
- Kähler form, 171
- Kähler identities, 194
- Kähler manifold, 173
- Kähler metric, 173
- Kähler potential, 188
- Kähler-Einstein metric, 223, 230
- Kählerian manifold, 201
- kernel sheaf, 110
- Killing form, 20
- Kobayashi distance, 232
- Kobayashi hyperbolic, 233
- Kobayashi-Royden metric, 234
- Kodaira conjecture
- Kodaira dimension, 132
- Kodaira embedding thm, 203
- Kodaira fibration, 153
- Kodaira surface, 142
- Kodaira vanishing thm, 203
- Kodaira-Nakano vanishing thm, 201
- Kodaira-Yau thm, 230
- Kummer surface, 152
- Künneth formula, 219

- L
- Laplace operator, 72
- Lefschetz decomposition, 196
- Lefschetz hyperplane section thm, 202
- Lefschetz thm on $(1, 1)$ classes, 196
- left multiplication, 16
- Leibniz' rule, 29
- length (of a curve), 49
- Levi Civita connection, 35
- Lie algebra, 18
 - of a Lie group, 18
- Lie bracket, 6
- Lie group, 16
- lifting property, 22
- line bundle,
 - associated to a divisor, 116
- linear connection, 31
- linear system, 117
- local 1-parameter gp of transformations, 6
- local coordinate neighborhood, 3
- local coordinate system, 3
- local defining function(s), 95, 97
- local ring, 96
- local syzygy, 125
- locally conformally flat, 47
- locally finite covering, 7
- locally finite union, 25
- locally free sheaf, 110
- locally irreducible, 209
- locally path connected, 22
- locally symmetric space, 68
- long exact sequence, 114, 125
- loop, 21
- Lu's inequality, 183

- M
- matrix exponential, 19
- matrix group, 17
- matrix of a metric, 158
- maximum principle, 85
- mean curvature, 46, 75
- meromorphic function, 99
- meromorphic map,
 - associated with a line bundle, 117
- metric, 29
- metric form, 171
- metric space, 23
- minimal complex surface, 134
- minimal curve, 51
- minimal degree rational curve, 216
- minimal geodesic, 63
- minimal model, 134
- minus one curve, 134
- Miyaoka-Yau inequality, 150
- Mok's thm, 215, 252, 253
- Mori's thm, 215
- moving part of a linear system, 131
- Myers' thm, 62

- N
- Nakai-Moishezon criterion, 205
- Nakano curvature tensor, 177
- natural bundle, 11
- natural frame, 11, 35
- naturality, 120
- Noether inequality, 150
- non-extendible Riemannian manifold, 74
- non-negative (p, p) -form, 170
- normal bundle, 47, 107
- normal coordinate, 36, 173
- normal neighborhood, 52
- normalization, 98
- numerically effective line bundle, 217

- O
- O'Neil's formula, 68

- open covering, 21
- open subset, 20
- orthogonal group, 17
- orientable manifold, 71
- orientation, 71
- Otsuki's lemma, 47

- P
- p -form, 12
- (p, q) -form, 106
- partition of unity, 7
- parallel field, 30
- parallel transport, 30
- path, 21
- path connected, 22
- pencil, 125
- Picard group, 116
- Picard lattice, 129
- Picard number, 129, 198
- piecewise smooth chain, 15
- piecewise smooth homology, 15
- pluriharmonic function, 101
- plurisubharmonic function, 101
- Poincaré duality thm, 16
- Poincaré lemma, 114, 123
- polarization formula, 161, 162
- pole, 58
- poles of a meromorphic function, 99
- positive cohomology class, 196, 205
- positive line bundle, 176, 201
- positive (p, p) -form, 170
- positive polynomial, 207
- positively curved Hermitian bundle, 176
- potential of a Kähler metric, 188
- Preissman's thm, 74, 78
- presheaf, 109
- primary Kodaira surface, 142
- prime divisor, 99
- principal curvature, 45
- principal divisor, 99
- projectified bundle, 107
- projection formula, 123
- projective manifold, 88
- proper map, 98
- proper transform, 125, 134
- properly discontinuous, 23
- properly elliptic surface, 143
- pull back connection, 159
- pull back sheaf, 123
- pure k -dimensional, 97

- Q
- quadric, 102
- quasi-negative, 247
- quasi-positive, 215
- quaternionic projective space, 208
- quotient bundle, 107
- quotient sheaf, 110
- quotient topology, 23

- R
- ramification locus, 204
- rank of a smooth map, 7
- rank of a symmetric space, 208
- rational curve, 87
- rational surface, 138
- rational equivalence, 99
- real projective space, 5
- reduced curve, 131
- reducible subvariety, 97
- regular (point, locus), 95, 97
- relative tangent bundle, 108
- resolution of singularity, 98
- Ricci curvature, 39
- Ricci form, 222
- Ricci tensor, 39, 180
 - first, second, third, 181
- Riemann extension thm, 103
- Riemann surface, 86
- Riemannian connection, 35
- Riemannian distance, 49
- Riemannian manifold, 34
- Riemannian metric, 34
- Riemannian submanifold, 67
- Riemannian symmetric space, 68
 - dual, 70
 - irreducible, 70
 - locally, 68
 - of (non) compact type, 70
- right multiplication, 16
- Ruan's coordinate, 188
- ruled surface, 138

- S
- scalar curvature, 39, 180
- Schubert cell, 89
- Schubert subvariety, 89
- Schur's polynomial, 207
- Schur's thm, 45
- Schwartz lemma for volume forms, 190
- second fundamental form, 47, 167
- second Ricci, 181
- second variation formula, 62
- secondary Kodaira surface, 142
- section, 8
- sectional curvature, 37
- semisimple Lie algebra, 20
- semisimple symmetric space, 70
- Serre duality thm, 115
- Seshadri's criterion, 217
- sheaf, 109
- sheaf associated to a presheaf, 124
- sheaf cohomology group, 113
- sheaf homomorphism, 110
- sheaf map, 110
- signature, 127
- simple Lie algebra, 20
- simple symmetric space, 70

- simplicial complex, 25
 - simply connected, 22
 - singular (point, locus), 95, 97
 - singular cochain, 14
 - singular cohomology group, 14
 - singular chain, 14
 - singular homology group, 14
 - singular simplex, 14
 - singularity, 98
 - Siu's strong rigidity thm, 240, 241
 - Siu-Sampson thm, 238
 - Siu-Yau thm, 214
 - smooth manifold, 3
 - smooth map, 3
 - soul thm, 74
 - space form, 66
 - special linear group, 17
 - special unitary group, 17
 - sphere, 4, 42
 - sphere thm, 74
 - splitting thm, 74, 79
 - stalk, 123
 - stalk space, 123
 - Stein factorization, 130
 - Stein manifold, 94
 - Stokes' thm, 15, 100
 - strongly negative, 239
 - strongly rigid, 240
 - structure equation, 41, 158, 172
 - structure sheaf, 111
 - subbundle, 13, 107
 - submanifold, 4, 88
 - subsheaf, 110
 - subspace, 21
 - subvariety, 96
 - support of a function, 7
 - surface in \mathbb{R}^3 , 45
 - developable, 46
 - minimal, 75
 - of revolution, 46
 - ruled, 46
 - symmetric product, 9, 10
 - symmetric space, 68
 - Euclidean, 70
 - irreducible, 70
 - semisimple, 70
 - simple, 70
 - symplectic group, 18
 - Synge thm, 76
- T
- tangent bundle, 11
 - tangent space, 5
 - tangent vector, 5
 - tautological line bundle, 107
 - tensor field, 11
 - tensor product, 9
 - third Ricci, 181
- Todd class, 122
 - topology, 20
 - topological space, 20
 - torsion (tensor), 32
 - of an almost complex structure, 91
 - torsion free, 32
 - torus, 74
 - transition function, 8
 - tree of rational curves, 131
 - triangulation, 26
 - two-point homogeneous space, 208
 - type I_m^0 singular fiber, 143
- U
- underlying smooth manifold, 89
 - unitary frame, 158
 - unitary group, 17
 - universal quotient bundle, 107
 - universal subbundle, 107
- V
- VII₀ surface, 139
 - vanishing and finiteness thm, 113
 - vector bundle, 8
 - vector field, 6
 - complete, 7
 - very ample divisor, 118
 - very ample line bundle, 132
 - volume form, 71, 171
- W
- wedge product, 12
 - Weierstrass polynomial, 103
 - Weierstrass preparation thm, 102
 - Weyl curvature tensor, 47, 225
 - Whitney sum formula, 120
 - Wirtinger's thm, 172
 - Wu metric, 235, 236
 - Wu's thm, 182
- Y
- Yau's conjecture, 236
 - Yau's maximum principle, 185
 - Yau's Schwartz lemma, 184
 - Yau's thm, 223
- Z
- zero locus, 117
 - zeroes of a meromorphic function, 99

Selected Titles in This Series

- 18 **Fangyang Zheng**, *Complex Differential Geometry*, 2000
- 17 **Lei Guo and Stephen S.-T. Yau**, *Editors*, *Lectures on Systems, Control, and Information*, 2000
- 16 **Rudi Weikard and Gilbert Weinstein**, *Editors*, *Differential Equations and Mathematical Physics*, 2000
- 15 **Ling Hsiao and Zhouping Xin**, *Editors*, *Some Current Topics on Nonlinear Conservation Laws*, 2000
- 14 **Jun-ichi Igusa**, *An Introduction to the Theory of Local Zeta Functions*, 2000
- 13 **Vasilios Alexiades and George Siopsis**, *Editors*, *Trends in Mathematical Physics*, 1999
- 12 **Sheng Gong**, *The Bieberbach Conjecture*, 1999
- 11 **Shinichi Mochizuki**, *Foundations of p -adic Teichmüller Theory*, 1999
- 10 **Duong H. Phong, Luc Vinet, and Shing-Tung Yau**, *Editors*, *Mirror Symmetry III*, 1999
- 9 **Shing-Tung Yau**, *Editor*, *Mirror Symmetry I*, 1998
- 8 **Jürgen Jost, Wilfrid Kendall, Umberto Mosco, Michael Röckner, and Karl-Theodor Sturm**, *New Directions in Dirichlet Forms*, 1998
- 7 **D. A. Buell and J. T. Teitelbaum**, *Editors*, *Computational Perspectives on Number Theory*, 1998
- 6 **Harold Levine**, *Partial Differential Equations*, 1997
- 5 **Qi-keng Lu, Stephen S.-T. Yau, and Anatoly Libgober**, *Editors*, *Singularities and Complex Geometry*, 1997
- 4 **Vyjayanthi Chari and Ivan B. Penkov**, *Editors*, *Modular Interfaces: Modular Lie Algebras, Quantum Groups, and Lie Superalgebras*, 1997
- 3 **Xia-Xi Ding and Tai-Ping Liu**, *Editors*, *Nonlinear Evolutionary Partial Differential Equations*, 1997
- 2.2 **William H. Kazez**, *Editor*, *Geometric Topology*, 1997
- 2.1 **William H. Kazez**, *Editor*, *Geometric Topology*, 1997
- 1 **B. Greene and S.-T. Yau**, *Editors*, *Mirror Symmetry II*, 1997

American
Mathematical
Society
www.ams.org

International
Press
www.intlpress.com

ISBN 0-8218-2960-2



9 780821 829608

AMSIP/18.S