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Studies in Advanced Mathematics

S.-T. Yau, Series Editor

Introduction to p-adic Analytic Number Theory

M. Ram Murty

American Mathematical Society • International Press

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https://doi.org/10.1090/amsip/027

AMS/IP
Studies in
Advanced
Mathematics

Volume 27

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2000 *Mathematics Subject Classification*. Primary 11–01, 11–02, 11E95, 11Sxx.

Library of Congress Cataloging-in-Publication Data

Murty, Maruti Ram.
Introduction to p-adic number theory / M. Ram Murty.
p. cm. — (AMS/IP studies in advanced mathematics, ISSN 1089-3288 ; v. 27)
Includes bibliographical references and index.
ISBN 0-8218-3262-X (alk. paper)
1. Number theory. 2. p-adic analysis. I. Title. II. Series.

QA241 M85 2002
512′.74—dc21 2002025584

AMS softcover ISBN 978-0-8218-4774-9

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10 9 8 7 6 5 4 3 2 1 14 13 12 11 10 09

One life runs through all like a continuous chain,
of which all these various forms represent the links,
link after link, extending almost infinitely,
but of the same one chain.

Vivekananda

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Preface

Following Hensel's discovery in 1897 of the p-adic numbers, Ostrowski's theorem of 1918 classifying all the possible norms that one can define on the rational numbers has been the starting point of what is now called the adelic perspective. Loosely speaking, this means that the rational numbers should not be thought of as merely a subset of the real numbers but rather as a subset of a spectrum of topological fields obtained by completing the rational number field with respect to each of the possible norms. This adelic perspective has been a unifying theme in number theory ever since its conception in the 20th century. From this vantage point, all completions of the rationals must be treated "equally." It would seem then that what is traditionally called analytic number theory looks at only one completion, and ignores the non-archimedean (or p-adic) completions. In the latter half of the 20th century, this restricted viewpoint was enlarged through the foundational work of Kubota and Leopoldt and later by Iwasawa who established much of the groundwork of a p-adic analytic number theory. Thus, the search for p-adic incarnations of the classical zeta and L-functions is of relatively recent origin and has been a useful motif in the study of special values of various L-functions and their arithmetic significance. This perspective has also been a fertile program of research, largely inspired by the method of analogy with the archimedean context. As such, it is an exciting program to discover to what extent p-adic analogues of classical theorems and in some cases, the classical methods, exist.

This monograph is a modest introduction to the p-adic universe. Its aim is to acquaint the non-expert to the basic ideas of the theory and to invite the novice to engage in a fecund field of research. It grew out of a course given to senior undergraduates and graduate students at Queen's University during the fall semester of 2000-2001. That course was based on notes of a shorter course given at the Harish-Chandra Research Institute on p-adic analytic number theory from December 22, 1999 till January 12, 2000.

The prerequisites have been kept to a minimum. My goal is to introduce the novice student to a beautiful chapter in number theory. A background in basic algebra, analysis, elementary number theory and at least one course in complex analysis should suffice to understand the first nine chapters. The last chapter requires a little bit more background and is intended to give some inspiration for studying the subject at a deeper level. There are more than a hundred exercises in the book and their level varies. Most of them are routine and the students in the course were able to keep pace by doing them.

I would like to thank Alina Cojocaru, Fernando Gouvêa. Hershy Kisilevsky, Yu-Ru Liu, Kumar Murty, D.S. Nagaraj, and Lawrence Washington for their careful and critical reading of preliminary versions of these lecture notes. I also thank the students and post-doctoral fellows at Queen's who participated in the seminar and course out of which the monograph was born.

M. Ram Murty
Kingston, Ontario, Canada
January 2002

Bibliography

[Ap] T. Apostol, Introduction to Analytic Number Theory, Undergraduate Texts in Mathematics, Springer-Verlag, 1976.

[Bo] R. Bojanic, A simple proof of Mahler's theorem on approximation of continuous functios of a p-adic variable by polynomials, *J. Number Theory*, **6** (1974)pp. 412-415.

[C] J. Cassels, Local Fields, *London Mathematical Society Student Texts*, **3**, Cambridge University Press., 1986.

[Cl] W. E. Clark, The Aryabhatiya of Aryabhata, University of Chicago Press, Chicago, Illinois, 1930.

[Co] R. Coleman, On the Galois groups of the exponential Taylor polynomials, *Enseign. Math.*, **33**(2) (1987), no. 3-4, 183-189.

[EM] J. Esmonde and M. Ram Murty, Problems in Algebraic Number Theory, GTM, Vol. 190, Springer-Verlag, 1999.

[FW] B. Ferrero and L. Washington, The Iwasawa invariant μ_p vanishes for abelian number fields, *Annals of Math.*, **109** (1979) pp. 377-395.

[G] F. Gouvêa, p-adic Numbers, Universitext, Springer, 1997.

[GS] D. Goldfeld and L. Szpiro, Bounds for the order of the Tate-Shafarevich group, *Comp. Math.*, **97** (1995), no.1-2, pp. 71-87.

[Gr] R. Greenberg, Iwasawa theory, Past and Present, in *Advanced Studies in Pure Mathematics*, **30** (2001), pp. 335-385.

[Iw] K. Iwasawa, On some properties of Γ-finite modules, *Annals of Math.*, **70**(2) (1959), pp. 530-561.

[J] G.J. Janusz, Algebraic Number Fields, Second Edition, *Graduate Texts in Mathematics*, **7**, American Math. Society, Providence, Rhode Island, 1996.

[K] N. Koblitz, p-adic Numbers, p-adic Analysis and Zeta Functions, Vol. 58, Springer-Verlag, 1977.

[K2] N. Koblitz, p-adic analysis: a short course on recent work, *London Math. Society Lecture Notes*, **46** (1980), Cambridge University Press, Cambridge, New York.

[Ki] Y. Kida, ℓ-extensions of CM fields and cyclotomic invariants, *Journal of Number Theory*, **12** (1980), pp. 519-528.

[L] S. Lang, Cyclotomic Fields, second edition, Springer-Verlag, 1990.

[MW] B.Mazur and A. Wiles, Class fields of abelian extensions of $\mathbb{Q}$, *Invent. Math.,* **76** (1984), no. 2, 179-330.

[MO] J. Oesterlé, Nouvelles approches du "théorème" de Fermat, *Sem. Bourbaki,* **694**, Astérisque 161-162, S.M.F., (1988), pp. 165-186.

[MaMu] L. Mai and M. Ram Murty, The Phragmén-Lindelöf theorem and modular elliptic curves, *Contemp. Math.,* **166** (1994), pp. 335-340, American Math. Society, Providence, R.I.

[M] M. Ram Murty, Problems in Analytic Number Theory, GTM/RIM, Vol. 206, Springer-Verlag, 2001.

[MM] M. Ram Murty and V. Kumar Murty, Non-vanishing of L-functions and Applications, *Progress in Mathematics,* **157**, Birkhaüser-Verlag, 1997.

[MR] M. Ram Murty and Marilyn Reece, A simple derivation of $\zeta(1-k) = -B_k/k$, *Functiones et Approximatio,* **28** (2000), 141-154.

[O] H. Osada, The Galois groups of the polynomials $x^n + ax^l + b$, *J. Number Theory,* **5** (1987), no. 2, 230-238.

[Ra] C. S. Rajan, On the size of the Shafarevich-Tate group of elliptic curves over function fields, *Comp. Math.,* **105** (1997), no. 1, pp. 29-41.

[Ro] A. Robert, A course in p-adic analysis, GTM, Vol. 198, Springer-Verlag, 2000.

[Se] E. Selmer, On the irreducibility of certain trinomials, *Math. Scand.,* **4** (1956), 287-302.

[Si] C. L. Siegel, Zu zwei Bemerkungen Kummers, *Nachr. Akad. Wiss., Göttingen, Math.-Phys. Kl.* (1964), no. 6, 51-57. (See also, *Gesammelte Abhandlungen,* Springer-Verlag, Berlin, 1966, Vol. 3, 436-442.)

[S] J. H. Silverman, Wieferich's criterion and the *abc*-conjecture, *J. Number Theory,* **30** (1988), no. 2, 226-237.

[W] L. Washington, Introduction to Cyclotomic Fields, GTM, Vol. 83, Second Edition, Springer-Verlag, 1997.

[W2] L. Washington, The non-p-part of the class number in a cyclotomic $\mathbb{Z}_p$-extension, *Inv. Math.,* **49** (1978), 87-97.

[W3] L. Washington, A family of cubic fields and zeros of 3-adic L-functions, *Journal of Number Theory,* **63** (1997), no. 2, 408-417.

[We] A. Weil, Number Theory, An Approach Through History, From Hammurapi to Legendre, Birkhaüser, 1984.

[Wi] A. Wiles, Modular elliptic curves and Fermat's last theorem, *Annals of Math.*, **141**(2) (1995), no. 3, 443-551.

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This book is an elementary introduction to *p*-adic analysis from the number theory perspective. With over 100 exercises included, it will acquaint the non-expert to the basic ideas of the theory and encourage the novice to enter this fertile field of research.

The main focus of the book is the study of *p*-adic *L*-functions and their analytic properties. It begins with a basic introduction to Bernoulli numbers and continues with establishing the Kummer congruences. These congruences are then used to construct the *p*-adic analog of the Riemann zeta function and *p*-adic analogs of Dirichlet's *L*-functions. Featured is a chapter on how to apply the theory of Newton polygons to determine Galois groups of polynomials over the rational number field. As motivation for further study, the final chapter introduces Iwasawa theory.

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ISBN 978-0-8218-4774-9

9 780821 847749

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