Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity

Weimin Han and Mircea Sofonea
Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity
This page intentionally left blank
Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity

Weimin Han and Mircea Sofonea
2000 Mathematics Subject Classification. Primary 74–02, 74M15, 74M10, 74S05, 74B05, 74B20, 74Cxx, 74Dxx; Secondary 65M06, 65M12, 65M15, 65M60, 65N12, 65N15, 65N30, 49J40.

Library of Congress Cataloging-in-Publication Data
Han, Weimin.
Quasistatic contact problems in viscoelasticity and viscoplasticity / Weimin Han and Mircea Sofonea.
p. cm. — (AMS/IP studies in advanced mathematics; v. 30)
Includes bibliographical references and index.

QA931 .H26 2002
620.1'1232—dc21 2002027716

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society and International Press. Requests for such permission should be addressed to the Acquisitions Department, American Mathematical Society, 201 Charles Street, Providence, Rhode Island 02904-2294, USA. Requests can also be made by e-mail to reprint-permission@ams.org.

© 2002 by the American Mathematical Society and International Press. All rights reserved.
The American Mathematical Society and International Press retain all rights except those granted to the United States Government.
Printed in the United States of America.

The paper used in this book is acid-free and falls within the guidelines established to ensure permanence and durability.
Visit the AMS home page at http://www.ams.org/
Visit the International Press home page at URL: http://www.intlpress.com/

10 9 8 7 6 5 4 3 2 1 07 06 05 04 03 02
Dedicated to

DAQING HAN, SUZHEN QIN
HUIDI TANG, ELIZABETH JING HAN
MICHAEL YUE HAN

and

In memory of
SEMINICA SOFO NEA
This page intentionally left blank
# Contents

Preface xi

List of Symbols xv

I Nonlinear Variational Problems and Numerical Approximation 1

1 Preliminaries of Functional Analysis 3
   1.1 Normed Spaces and Banach Spaces .................. 3
   1.2 Linear Operators and Linear Functionals .......... 8
   1.3 Hilbert Spaces .................................... 13
   1.4 Convex Functions .................................. 17
   1.5 Banach Fixed-point Theorem ........................ 21

2 Function Spaces and Their Properties 25
   2.1 The Spaces $C^m(\bar{\Omega})$ and $L^p(\Omega)$ ....... 25
   2.2 Sobolev Spaces .................................... 29
   2.3 Spaces of Vector-valued Functions .................. 37

3 Introduction to Finite Difference and Finite Element Approximations 43
   3.1 Finite Difference Approximations .................... 43
   3.2 Basis of the Finite Element Approximation ........... 45
   3.3 Finite Element Interpolation Error
      Estimates ....................................... 53
   3.4 Finite Element Analysis of Linear Elliptic Boundary Value Problems ................... 54

4 Variational Inequalities 59
   4.1 Elliptic Variational Inequalities ................... 59
   4.2 Approximation of Elliptic Variational
      Inequalities ..................................... 65
   4.3 An Evolutionary Variational Inequality ............... 68
   4.4 Semi-discrete Approximations of the Evolutionary Variational Inequality ............ 76
   4.5 Fully Discrete Approximations of the Evolutionary Variational
      Inequality ..................................... 78

Bibliographical Notes 89
II Mathematical Modelling in Contact Mechanics 91

5 Preliminaries of Contact Mechanics of Continua 93
   5.1 Kinematics of Continua ........................................ 93
   5.2 Dynamics of Continua .......................................... 98
   5.3 Physical Setting of Contact Problems ...................... 102
   5.4 Contact Boundary Conditions and Friction Laws ......... 104

6 Constitutive Relations in Solid Mechanics 113
   6.1 Physical Background and Experiments ..................... 113
   6.2 Constitutive Relations in Elasticity ....................... 121
   6.3 Constitutive Relations in Viscoelasticity ............... 125
   6.4 Constitutive Relations in Viscoplasticity ............... 133

7 Background on Variational and Numerical Analysis in Contact Mechanics 141
   7.1 Function Spaces in Solid Mechanics ....................... 141
   7.2 Semi-discrete and Fully Discrete Approximations .......... 152
   7.3 Convergence under Basic Solution Regularity .......... 156
   7.4 Some Inequalities ............................................ 162

8 Contact Problems in Elasticity 167
   8.1 Frictionless Contact Problems ............................. 167
   8.2 Numerical Analysis of the Frictionless Contact Problems . 173
   8.3 Quasistatic Frictional Contact Problems ................ 177
   8.4 Numerical Analysis of Quasistatic Frictional Contact Problems . 181
   8.5 Numerical Examples ........................................... 184

Bibliographical Notes 189

III Contact Problems in Viscoelasticity 191

9 A Frictionless Contact Problem 193
   9.1 Problem Statement ........................................... 193
   9.2 An Existence and Uniqueness Result ...................... 195
   9.3 Numerical Approximations ................................. 197
   9.4 Dual Formulation ............................................ 202

10 Bilateral Contact with Slip Dependent Friction 207
   10.1 Problem Statement .......................................... 207
   10.2 An Existence and Uniqueness Result .................... 209
   10.3 Semi-discrete Approximation ............................. 214
   10.4 Fully Discrete Approximations .......................... 218
   10.5 Dual Formulation .......................................... 222
CONTENTS

11 Frictional Contact with Normal Compliance 227
  11.1 Problem Statement ........................................ 227
  11.2 An Abstract Problem and its
       Well-posedness ........................................ 230
  11.3 Semi-discrete Approximation ............................. 234
  11.4 Fully Discrete Approximation ........................... 237
  11.5 Applications to the Contact Problem .................... 239
  11.6 Continuous Dependence with Respect to Contact Conditions 243
  11.7 Numerical Examples .................................... 246

12 Frictional Contact with Normal Damped Response 255
  12.1 Problem Statement ........................................ 255
  12.2 An Abstract Problem and its
       Well-posedness ........................................ 257
  12.3 Semi-discrete Approximation of the
       Abstract Problem ...................................... 262
  12.4 Fully Discrete Approximation of the Abstract Problem 264
  12.5 Applications to the Contact Problem .................... 267
  12.6 Two Field Variational Formulations ...................... 271
  12.7 Numerical Examples .................................... 273

13 Other Viscoelastic Contact Problems 285
  13.1 Bilateral Contact with Nonlocal Coulomb Friction Law ..... 285
  13.2 Bilateral Contact with Friction and Wear ................ 289
  13.3 Contact with Normal Compliance, Friction and Wear .... 292
  13.4 Contact with Dissipative Frictional Potential .......... 296

Bibliographical Notes 305

IV Contact Problems in Viscoelasticity 307

14 A Signorini Contact Problem 309
  14.1 Problem Statement ........................................ 309
  14.2 Existence and Uniqueness Results ....................... 311
  14.3 Some Properties of the Solution ....................... 316

15 Frictionless Contact with Dissipative Potential 321
  15.1 Problem Statement and Variational Analysis .......... 321
  15.2 Semi-discrete Approximation ........................... 324
  15.3 Fully Discrete Approximation ........................... 327
  15.4 The Signorini Contact Problem .......................... 330
  15.5 A Frictionless Contact Problem with Normal Compliance 333
  15.6 A Convergence Result .................................. 335
  15.7 Numerical Examples .................................... 340
CONTENTS

16 Frictionless Contact between Two Viscoplastic Bodies 347
   16.1 Problem Statement .............................................. 347
   16.2 Unique Solvability and Properties of the Solution ............ 350
   16.3 Semi-discrete Approximation .................................... 352
   16.4 Fully Discrete Approximation .................................. 357
   16.5 Numerical Examples ............................................. 360

17 Bilateral Contact with Tresca’s Friction Law 365
   17.1 Problem Statement .............................................. 365
   17.2 Existence and Uniqueness Results ............................... 368
   17.3 Some Properties of the Solution ................................. 376
   17.4 Semi-discrete Approximation .................................... 379
   17.5 Fully Discrete Approximation .................................. 384
   17.6 Convergence of the Fully Discrete Scheme ...................... 390

18 Other Viscoplastic Contact Problems 397
   18.1 Contact with Simplified Coulomb’s Friction Law ............... 397
   18.2 Contact with Dissipative Frictional Potential .................. 399
   18.3 Stress Formulation in Perfect Plasticity ....................... 404
   18.4 A Frictionless Contact Problem for Materials with Internal State Variable ......................... 414

Bibliographical Notes 421

Bibliography 423

Index 439
Preface

Phenomena of contact between deformable bodies or between deformable and rigid bodies abound in industry and everyday life. Contact of braking pads with wheels, tires with roads, pistons with skirts are just a few simple examples. Common industrial processes such as metal forming, metal extrusion, involve contact evolutions. Because of the importance of contact processes in structural and mechanical systems, a considerable effort has been made in its modeling and numerical simulations. The engineering literature concerning this topic is extensive. Owing to their inherent complexity, contact phenomena are modeled by nonlinear evolutionary problems that are difficult to analyze.

In early mathematical publications it was invariably assumed that the deformable bodies were linearly elastic and the processes were static. However, a long time ago it was recognized the need to consider contact problems involving viscoelastic and viscoplastic materials as well as a large variety of contact and frictional boundary conditions, which lead to time dependent models. The mathematical theory of contact problems, that can predict reliably the evolution of the contact process in different situations and under various conditions, is emerging currently. It deals with rigorous modeling of the contact phenomena, based on the fundamental physical principles, as well as with the variational and numerical analysis of the models. A thorough treatment of contact problems require knowledge from functional analysis, modern partial differential equations, numerical approximations and error analysis.

The purpose of this book is to introduce the reader to a mathematical theory of contact problems involving deformable bodies. The contents cover the mechanical modeling, mathematical formulations, variational analysis, and the numerical solution of the associated formulations. Our intention is to give a complete treatment of some contact problems by presenting arguments and results in modeling, analysis, and numerical simulations.

In the book we treat quasistatic contact processes in the infinitesimal strain theory. Quasistatic processes arise when the applied forces vary slowly in time, and therefore the system response is relatively slow so that the inertial terms in the equations of motion can be neglected. We model the material behavior with elastic, viscoelastic or viscoplastic constitutive laws; some of our results are extended to materials with internal state variables and to perfectly plastic materials. The contact is modeled with various conditions, including Signorini nonpenetration condition, normal compliance and normal damped response conditions. The friction is modeled with versions of Coulomb’s and Tresca’s friction laws or with laws involving a dissipative frictional potential. We also consider problems with friction and wear and we use a version of the Archard law to model the evolution of wear.
Variational analysis of the models includes existence and uniqueness results of weak solutions as well as results of continuous dependence of the solution on the data and parameters. Links between different mechanical models are discussed; for example, elasticity as a limiting case of viscoelasticity, perfect plasticity as a limiting case of viscoplasticity, Signorini nonpenetration condition as a limiting case of the normal compliance contact condition. In carrying out the variational analysis we systematically use results on elliptic and evolutionary variational inequalities, convex analysis, nonlinear equations with monotone operators and fixed points of operators.

Two kinds of approximation schemes are introduced and analyzed. When only the spatial variables are discretized, we obtain semi-discrete schemes. If both the spatial and temporal variables are discretized, we arrive at fully discrete schemes. For both kinds of schemes we prove existence and uniqueness results. We show convergence of the discrete solutions under the basic solution regularity available from the well-posedness results of the variational problems. We also present optimal order error estimates under additional regularity assumption on the solution.

To demonstrate the performance of the numerical schemes, a number of numerical simulations are discussed. The test problems range from one to three dimensional geometries. The finite element method is used to discretize the spatial domain and finite differences are used for the time derivatives. We describe in the book numerical results in the study of some model contact problems for elastic, viscoelastic and viscoplastic materials.

The book is intended to be self-contained and accessible to a large number of readers. It is divided into four parts, as described in the following.

Part I is devoted to the basic notions and results which are fundamental to the development later in this book. We review here the background on functional analysis, function spaces, finite difference approximations and finite element method. Then we apply these results in the study of elliptic and evolutionary variational inequalities. The material presented in this part is self-contained. It does not need any knowledge in Contact Mechanics and could be aimed at graduate students and researchers interested in a general treatment of variational problems and their numerical approximations.

Part II presents preliminary material in Contact Mechanics. We summarize here basic notions and general principles of Mechanics of Continua. Then we introduce contact boundary conditions with or without friction as well as constitutive laws which are used in the rest of the book. We also present preliminary results on variational and numerical analysis in contact problems and we apply these results in the study of models involving elastic bodies. The material in Part II is aimed at those readers who are interested in the mechanical background of contact problems and mathematical theory of some contact problems in elasticity.

Parts III and IV represent the main parts of the book. They deal with the study of quasistatic problems for Kelvin-Voigt viscoelastic materials and rate-type viscoplastic materials, respectively. These parts are written based on our original research. We consider here a number of problems with various
contact and frictional or frictionless boundary conditions, for which we provide variational analysis and numerical approximations. For some of the contact problems, we include results of numerical simulations to show the performance of the numerical schemes. These two parts of the book would interest mainly researchers for an in-depth knowledge of the mathematical theory of quasistatic contact problems.

The list of the references at the end of the book is by no means exhaustive. It only includes papers or books that were used for or are directly connected with the subjects treated in this book. Each part is concluded with a section entitled Bibliographical Notes that discusses references on the principal results treated, as well as information on important topics related to, but not included in, the body of the text.

Each of the four parts of the book is divided into several chapters. All the chapters are numbered consecutively. Mathematical relations (equalities or inequalities) are numbered by chapter and their order of occurrence. For example, (5.3) is the third numbered mathematical relation in Chapter 5. Definitions, examples, problems, theorems, lemmas, corollaries, propositions and remarks are numbered consecutively within each chapter. For example, in Chapter 10, Problem 10.1 is followed by Theorem 10.2.

The present book is the result of three years of cooperation between the two authors. In writing it we have drawn on the results of our joint collaboration with numerous colleagues and friends to whom we address our thanks. We express our gratitude to Professor Meir Shillor for our beneficial cooperation as well as for the interesting discussions on the models treated in the book. We particularly thank Dr. J.R. Fernández-García who provided the numerical simulations included in Parts III and IV of the book. We extend our thanks to Dr. M. Barboteau who realized the numerical results in Part II of the book. We especially thank Professor Shing-Tung Yau for his support and encouragement of our work.

The work of W.H. was supported by NSF/DARPA under Grant DMS-9874015, James Van Allen Natural Science Fellowship, Faculty Development Award at the University of Iowa, and K.C. Wong Education Foundation. Part of the book manuscript was prepared when W.H. was visiting the State Key Laboratory of Scientific and Engineering Computing, Chinese Academy of Sciences, during the summer of 2000 through the support of the K.C. Wong Education Foundation. He particularly thanks Professor Lie-heng Wang for the warm hospitality.

W.H. 
Iowa City

M.S. 
Perpignan
List of Symbols

Sets

\( \mathbb{N} \): the set of positive integers
\( \mathbb{Z}_+ \): the set of non-negative integers
\( \mathbb{R} \): the real line
\( \bar{\mathbb{R}} = \mathbb{R} \cup \{\pm \infty\} \): extended real line
\( \mathbb{R}_+ \): the set of non-negative numbers
\( \mathbb{R}^d \): the \( d \)-dimensional Euclidean space
\( S^d \): the space of second order symmetric tensors on \( \mathbb{R}^d \)
\( Q^d_+ \): the set of orthogonal matrices of order \( d \) with determinant 1
\( S^d_d \): the unit sphere in \( \mathbb{R}^d \)
\( \Omega \): an open, bounded, connected set in \( \mathbb{R}^d \) with a Lipschitz boundary \( \Gamma \)
\( \Gamma \): the boundary of the domain \( \Omega \), that is decomposed as \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \) with \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) relatively open with respect to \( \Gamma \)
\( \Gamma_1 \): the part of the boundary where displacement condition is specified; \( \text{meas}(\Gamma_1) > 0 \) is assumed throughout the book
\( \Gamma_2 \): the part of the boundary where traction condition is specified
\( \Gamma_3 \): the part of the boundary for contact
\( \bar{I} = [0, T] \): time interval of interest
\( I = (0, T) \)
LIST OF SYMBOLS

Operators

\( \varepsilon: \) deformation operator, i.e. \( \varepsilon(u) = (\varepsilon_{ij}(u)) \),
\( \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}) \) (page 97)

\( \text{Div:} \) divergence operator, i.e. \( \text{Div} \sigma = \sigma_{ij,j} \) (page 100)

\( \Pi^h: \) finite element interpolation operator (page 52)

\( \mathcal{P}_K: \) projection operator onto a set \( K \) (page 16)

\( \mathcal{P}^h: \) finite element projection operator (pages 77, 181)

\( \mathcal{P}_{Q^h}: \) finite element projection on \( Q^h \) (page 154)

Function spaces

\( L^p(\Omega): \) the Lebesgue space of \( p \)-integrable functions, with the usual modification if \( p = \infty \) (page 28)

\( C^m(\Omega): \) the space of functions whose derivatives up to and including order \( m \) are continuous up to the boundary \( \Gamma \) (page 26)

\( C_0^\infty(\Omega): \) the space of infinitely differentiable functions with compact support in \( \Omega \) (page 26)

\( W^{k,p}(\Omega): \) the Sobolev space of functions whose weak derivatives of orders less than or equal to \( k \) are \( p \)-integrable on \( \Omega \) (page 30)

\( H^k(\Omega) \equiv W^{k,2}(\Omega) \) (page 30)

\( W_0^{k,p}(\Omega): \) the closure of \( C_0^\infty(\Omega) \) in \( W^{k,p}(\Omega) \) (page 30)

\( H_0^k(\Omega) \equiv W_0^{k,2}(\Omega) \) (page 30)

\( H^{-1}(\Omega): \) the dual of \( H_0^1(\Omega) \) (page 34)

\( H^{\frac{1}{2}}(\Gamma): \) a Sobolev space on \( \Gamma \), defined as the range of the trace operator on \( H^1(\Omega) \) (page 36)

\( H^{-\frac{1}{2}}(\Gamma): \) the dual of \( H^{\frac{1}{2}}(\Gamma) \) (page 36)

\( H^d_\Gamma = H^{\frac{1}{2}}(\Gamma)^d \) (page 143)

\( H_\Gamma^*: \) dual of \( H_\Gamma \) (page 144)

\( H = \{ \mathbf{v} = (v_1, \ldots, v_d)^T : v_i \in L^2(\Omega), 1 \leq i \leq d \} = L^2(\Omega)^d , \)
inner product \( (\mathbf{u}, \mathbf{v})_H = \int_\Omega u_i(x) v_i(x) \, dx \) (page 142)

\( Q = \{ \mathbf{\tau} = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega), 1 \leq i, j \leq d \} = L^2(\Omega)^{d \times d} , \)
inner product \( (\mathbf{\sigma}, \mathbf{\tau})_Q = \int_\Omega \sigma_{ij}(x) \tau_{ij}(x) \, dx \) (page 142)

\( Q_1 = \{ \mathbf{\tau} \in Q : \text{Div} \mathbf{\tau} \in H \} , \)
inner product \( (\mathbf{\sigma}, \mathbf{\tau})_{Q_1} = (\mathbf{\sigma}, \mathbf{\tau})_Q + (\text{Div} \mathbf{\sigma}, \text{Div} \mathbf{\tau})_H \) (page 145)

\( V = \{ \mathbf{v} \in H^1(\Omega)^d : \mathbf{v} = 0 \text{ a.e. on } \Gamma_1 \} , \)
in inner product \( (\mathbf{u}, \mathbf{v})_V = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_Q \) (page 144)
LIST OF SYMBOLS

$V_1 = \{ v \in V : v_\nu = 0 \text{ a.e. on } \Gamma_3 \}$ with the inner product $(u,v)_V$ (page 144)

$V_2 = \{ v \in V : v_\nu \leq 0 \text{ a.e. on } \Gamma_3 \}$ with the inner product $(u,v)_V$ (page 144)

$X$: a Hilbert space or its subset with inner product $(\cdot,\cdot)_X$, or a Banach space or its subset with norm $\| \cdot \|_X$

$C^m(I;X) = \{ v \in C(I;X) : v^{(j)} \in C(I;X), j = 1, \ldots, m \}$ (page 37)

$L^p(I;X) = \{ v : I \to X \text{ measurable} : \|v\|_{L^p(I;X)} < \infty \}$ (page 38)

$W^{k,p}(I;X) = \{ v \in L^p(0,T;X) : \|v^{(j)}\|_{L^p(I;X)} < \infty \ \forall \ j \leq k \}$ (page 39)

$H^k(I;X) \equiv W^{k,2}(I;X)$ (page 40)

Other symbols

$d$: a positive integer taking the values 1, 2, 3 in applications

c: a generic positive constant

$r_+ = \max\{0, r\}$: positive part of $r$

$\nu$: unit outward normal on the boundary of $\Omega$

$(\cdot,\cdot)$: the duality pairing between $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$ (page 36)

$(\cdot,\cdot)_\Gamma$: the duality pairing between $H_\Gamma$ and $H^0_\Gamma$ (page 144)

$\forall$: for any

$\exists$: there exist(s)

$\overline{A}$: closure of the set $A$

$\text{int} A$ or $\overset{\circ}{A}$: interior of the set $A$

$\partial A$: boundary of the set $A$

$\delta_{ij}$: the Kronecker delta

s.t.: such that

a.e.: almost everywhere

iff: if and only if

$O(h)$: for some constant $c > 0$ independent of $h$ such that $|O(h)| \leq ch$

$\Delta w_n = w_n - w_{n-1}$: backward difference

$\delta_n w_n = \Delta w_n/k_n$: backward divided difference
Bibliography


[35] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contrac-


[39] O. Chau, W. Han and M. Sofonea, Analysis and aproximation of a vis-
coelastic contact problem with slip dependent friction, *Dynamics of Con-


[41] O. Chau, D. Motreanu and M. Sofonea, Quasistastic frictional problems for elastic and viscoelastic materials, to appear in *Applications of Mathe-
matics*.


BIBLIOGRAPHY


BIBLIOGRAPHY


[191] A. Signorini, Sopra a une questioni di elastostatica, Atti della Società Italiana per il Progresso delle Scienze, 1933.


BIBLIOGRAPHY


Index

absolute continuity, 40
acceleration, 95

balance law of angular momentum, 100
balance law of mass, 99
balance law of momentum, 100
Banach fixed-point theorem, 21
Banach space, 7
bidual, 12
bilinear form, 63
    bounded, 63
    continuous, 63
    elliptic, 63
    symmetric, 63
Bochner integrable, 38
bounded operator, 9
bulk modulus, 123
Burgers model, 128

Céa’s inequality, 56
    generalization of, 66, 67
Cartesian product space, 4
Cauchy sequence, 6
Cauchy stress tensor, 101
Cauchy–Lipschitz theorem, 202
Cauchy–Schwartz inequality, 14, 28
closed set, 6
    weakly, 12
closure, 7
compact support, 26
compliance tensor, 122
conjugate exponent, 11
constitutive law, 104
contact condition, 106
    bilateral, 106
    normal compliance, 107
    normal damped response, 107
    Signorini, 106
contact problem in elasticity
    bilateral with Tresca’s friction, 177
    frictionless with deformable support, 172
    Signorini frictionless, 167
    simplified Coulomb’s friction, 179
contact problem in perfect plasticity, 403
    bilateral with Tresca’s friction, 412
    frictionless with normal damped response, 412
    normal damped response with Tresca’s friction, 413
contact problem in viscoelasticity
    bilateral with friction and wear, 289
    bilateral with nonlocal Coulomb friction, 285
    bilateral with slip dependent friction, 207
    bilateral with Tresca’s friction, 302
    bilateral with viscous friction, 302
    frictional with normal damped response, 255
    frictional with normal compliance, 227
    normal damped response with power-law friction, 304
    normal damped response with Tresca’s friction, 304
    Signorini frictionless, 193
    viscous contact with power-law friction, 303
    viscous contact with Tresca’s friction, 303
    with dissipative frictional potential, 296
    with normal compliance, friction and wear, 292
contact problem in viscoplasticity
    bilateral with Tresca’s friction,
frictionless for materials with internal state variable, 414
frictionless with dissipative potential, 321
frictionless with normal compliance, 333
Signorini frictionless, 309, 330
unilateral frictionless between two bodies, 347
with dissipative frictional potential, 399
with simplified Coulomb’s friction, 397
continuous embedding, 33
continuous medium, 93
continuous operator, 9
continuum, 93
convergence, 6, 57
   strong, 6
   weak, 11
   weak *, 12
convex combination, 8
convex function, 17
   strictly, 17
convex set, 8
creep, 117
current configuration, 94
deformation gradient, 95
deformed configuration, 94
dense subset, 8
density of smooth functions, 148
derivative, 37
   strong, 37
   weak, 39
difference
   backward, 43
   centered, 43
   forward, 43
displacement, 95
   normal, 105
   tangential, 105
dual formulation, 202, 222
dual space, 11
dynamic processes, 102
effective domain, 18
elastic deformation, 116
elasticity operator, 132
elasticity tensor, 122
elastoplastic model, 117
elliptic variational inequality, 59
   first kind, 59
   second kind, 59
epigraph, 18
equation of equilibrium, 102
equation of motion, 102
error estimate, 57
Euclidean norm, 4
Euclidean space, 4
finite element interpolant, 52
finite element projection, 77
finite element space, 50
finite elements
   affine-equivalent, 49
finite strain tensor, 96
first Piola-Kirchhoff stress tensor, 100
first Piola-Kirchhoff stress vector, 100
fixed point, 21
friction condition, 106
   Coulomb, 110
   generalized Coulomb, 110
   nonlocal Coulomb, 111
   slip dependent Tresca, 109
   Tresca, 109
   viscous, 111
friction force, 106
frictionless condition, 108
Frobenius norm, 4
fully discrete approximation, 78
functional, 9
   linear continuous, 11
Gâteaux derivative, 20
generalized variational lemma, 29
generalized Voigt model, 128
global interpolation, 52
Green-Saint Venant tensor, 96
Hölder inequality, 28
Hencky material, 124
INDEX

Hilbert space, 14
Hooke’s law, 121
indicator function, 19
inelastic deformation, 116
infinitesimal deformation, 97
infinitesimal strain tensor, 97
inner product, 13
inner product space, 13
internal state variable, 138
interpolation error estimate, 53

Jacobian, 94
Kelvin-Voigt model, 126
kinematics of continua, 93
Korn’s inequality, 144

Lamé coefficients, 123
Lax-Milgram Lemma, 64
Lebesgue’s theorem, 29
limit, 6
  weak, 11
  weak *, 12
linear space, 3
linearized strain tensor, 97
Lipschitz continuous function, 27
Lipschitz continuous operator, 9
Lipschitz domain, 31
local interpolation, 51
locally $p$-integrable, 29
lower semicontinuous (l.s.c.) function, 18

mass density, 98
material
  anisotropic, 123
  homogeneous, 123
  isotropic, 123
  nonhomogeneous, 123
material point, 94
Maxwell model, 126
  generalized, 126
mesh, 46
mesh parameter, 47
motion, 94
multi-index, 25

norm, 3
  equivalent, 5
  Euclidean, 4
  Frobenius, 4
  operator, 10
normed space, 4
  complete, 7
  reflexive, 12
  separable, 13

operator, 8
  bounded, 9
  compact, 13
  continuous, 9
  domain, 8
  linear, 9
  Lipschitz continuous, 9
  maximal monotone, 195
  monotone, 15, 195
  range, 9
  strictly monotone, 15
  strongly monotone, 15
orthogonal, 15

partition, 46
perfectly plastic model, 134, 138
Perzyna’s law, 136
Poisson’s ratio, 123
Prandtl–Reuss flow rule, 138
primal formulation, 202, 222
projection, 16
projection operator, 16
proper, 17
proximal element, 61
proximity operator, 61
quasistatic processes, 102
quasivariational inequality, 272

reference configuration, 94
reference element, 46
reflexive normed space, 12
relaxation, 118
Riesz representation theorem, 15
rigid body motion, 95
rigid-plastic model, 117

scheme
   backward, 44
   Crank-Nicolson, 44
   forward, 44
   generalized mid-point, 45
semi-discrete approximation, 76
semi-norm, 5
separable normed space, 13
small strain tensor, 97
Sobolev space, 29
   embedding, 32
   of integer order, 30
   of non-integer order, 35
Sokolowski model, 134
standard viscoelastic model, 126
step-size, 43
strain
   elastic, 116
   plastic, 116
strength limit, 116
stress
   normal, 106
   tangential, 106
stress tensor, 102
stress vector, 102
stress-strain diagram, 113
subdifferentiable function, 19
subdifferential, 19
subgradient, 19
support, 26
support functional, 20

trace, 33, 143, 146
trace operator, 34, 143, 146
triangulation, 46
   regular, 47
tribology, 104

undeformed configuration, 94

velocity, 95
viscoelastic constitutive law, 132
viscoelastic phenomenon, 118
viscoplastic constitutive law, 133, 135

viscosity operator, 132, 194
von Mises function, 137

weak * convergence, 12
weak * limit, 12
weak convergence, 11
weak derivative, 29
weak formulation, 55
weak limit, 11
weakly closed set, 12
weakly lower semicontinuous (w.l.s.c.)
   function, 18

yield condition, 137
yield function, 137
yield limit, 116
Young’s modulus, 121, 123
Titles in This Series

30  Weiman Han and Mircea Sofonea, Quasistatic Contact Problems in Viscoelasticity and Viscoelasticity, 2002
29  Shuxing Chen and S.-T. Yau, Editors, Geometry and Nonlinear Partial Differential Equations, 2002
28  Valentin Afraimovich and Sze-Bi Hsu, Lectures on Chaotic Dynamical Systems, 2002
27  M. Ram Murty, Introduction to $p$-adic Analytic Number Theory, 2002
26  Raymond Chan, Yue-Kuen Kwok, David Yao, and Qiang Zhang, Editors, Applied Probability, 2002
25  Donggao Deng, Daren Huang, Rong-Qing Jia, Wei Lin, and Jian Zhong Wong, Editors, Wavelet Analysis and Applications, 2002
24  Jane Gilman, William W. Menasco, and Xiao-Song Lin, Editors, Knots, Braids, and Mapping Class Groups—Papers Dedicated to Joan S. Birman, 2001
22  Carlos Berenstein, Der-Chen Chang, and Jingzhi Tie, Laguerre Calculus and Its Applications on the Heisenberg Group, 2001
21  Jürgen Jost, Bosonic Strings: A Mathematical Treatment, 2001
19  So-Chin Chen and Mei-Chi Shaw, Partial Differential Equations in Several Complex Variables, 2001
18  Fangyang Zheng, Complex Differential Geometry, 2000
17  Lei Guo and Stephen S.-T. Yau, Editors, Lectures on Systems, Control, and Information, 2000
16  Rudi Weikard and Gilbert Weinstein, Editors, Differential Equations and Mathematical Physics, 2000
15  Ling Hsiao and Zhourping Xin, Editors, Some Current Topics on Nonlinear Conservation Laws, 2000
14  Jun-ichi Igusa, An Introduction to the Theory of Local Zeta Functions, 2000
13  Vasilios Alexiades and George Siopsis, Editors, Trends in Mathematical Physics, 1999
12  Sheng Gong, The Bieberbach Conjecture, 1999
11  Shinichi Mochizuki, Foundations of $p$-adic Teichmüller Theory, 1999
10  Duong H. Phong, Luc Vinet, and Shing-Tung Yau, Editors, Mirror Symmetry III, 1999
9  Shing-Tung Yau, Editor, Mirror Symmetry I, 1998
8  Jürgen Jost, Wilfrid Kendall, Umberto Mosco, Michael Röckner, and Karl-Theodor Sturm, New Directions in Dirichlet Forms, 1998
7  D. A. Buell and J. T. Teitelbaum, Editors, Computational Perspectives on Number Theory, 1998
6  Harold Levine, Partial Differential Equations, 1997
5  Qi-keng Lu, Stephen S.-T. Yau, and Anatoly Libgober, Editors, Singularities and Complex Geometry, 1997
4  Vyjayanthi Chari and Ivan B. Penkov, Editors, Modular Interfaces: Modular Lie Algebras, Quantum Groups, and Lie Superalgebras, 1997
3  Xia-Xi Ding and Tai-Ping Liu, Editors, Nonlinear Evolutionary Partial Differential Equations, 1997

2.2 William H. Kazez, Editor, Geometric Topology, 1997
2.1 William H. Kazez, Editor, Geometric Topology, 1997
TITLES IN THIS SERIES

1  B. Greene and S.-T. Yau, Editors, Mirror Symmetry II, 1997
Phenomena of contact between deformable bodies or between deformable and rigid bodies abound in industry and in everyday life. A few simple examples are brake pads with wheels, tires on roads, and pistons with skirts. Common industrial processes such as metal forming and metal extrusion involve contact evolutions. Because of the importance of contact processes in structural and mechanical systems, considerable effort has been put into modeling and numerical simulations.

This book introduces readers to a mathematical theory of contact problems involving deformable bodies. It covers mechanical modeling, mathematical formulations, variational analysis, and the numerical solution of the associated formulations. The authors give a complete treatment of some contact problems by presenting arguments and results in modeling, analysis, and numerical simulations.

Variational analysis of the models includes existence and uniqueness results of weak solutions, as well as results of continuous dependence of the solution on the data and parameters. Also discussed are links between different mechanical models.

In carrying out the variational analysis, the authors systematically use results on elliptic and evolutionary variational inequalities, convex analysis, nonlinear equations with monotone operators, and fixed points of operators.

Prerequisites include basic functional analysis, variational formulations of partial differential equation problems, and numerical approximations. The text is suitable for graduate students and researchers in applied mathematics, computational mathematics, and computational mechanics.