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Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity

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Dedicated to

DAQING HAN, SUZHEN QIN
HUIDI TANG, ELIZABETH JING HAN
MICHAEL YUE HAN

and

In memory of
SEMINICA SOFONEA

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Contents

Preface

Phenomena of contact between deformable bodies or between deformable and rigid bodies abound in industry and everyday life. Contact of braking pads with wheels, tires with roads, pistons with skirts are just a few simple examples. Common industrial processes such as metal forming, metal extrusion, involve contact evolutions. Because of the importance of contact processes in structural and mechanical systems, a considerable effort has been made in its modeling and numerical simulations. The engineering literature concerning this topic is extensive. Owing to their inherent complexity, contact phenomena are modeled by nonlinear evolutionary problems that are difficult to analyze.

In early mathematical publications it was invariably assumed that the deformable bodies were linearly elastic and the processes were static. However, a long time ago it was recognized the need to consider contact problems involving viscoelastic and viscoplastic materials as well as a large variety of contact and frictional boundary conditions, which lead to time dependent models. The mathematical theory of contact problems, that can predict reliably the evolution of the contact process in different situations and under various conditions, is emerging currently. It deals with rigorous modeling of the contact phenomena, based on the fundamental physical principles, as well as with the variational and numerical analysis of the models. A thorough treatment of contact problems require knowledge from functional analysis, modern partial differential equations, numerical approximations and error analysis.

The purpose of this book is to introduce the reader to a mathematical theory of contact problems involving deformable bodies. The contents cover the mechanical modeling, mathematical formulations, variational analysis, and the numerical solution of the associated formulations. Our intention is to give a complete treatment of some contact problems by presenting arguments and results in modeling, analysis, and numerical simulations.

In the book we treat quasistatic contact processes in the infinitesimal strain theory. Quasistatic processes arise when the applied forces vary slowly in time, and therefore the system response is relatively slow so that the inertial terms in the equations of motion can be neglected. We model the material behavior with elastic, viscoelastic or viscoplastic constitutive laws; some of our results are extended to materials with internal state variables and to perfectly plastic materials. The contact is modeled with various conditions, including Signorini nonpenetration condition, normal compliance and normal damped response conditions. The friction is modeled with versions of Coulomb's and Tresca's friction laws or with laws involving a dissipative frictional potential. We also consider problems with friction and wear and we use a version of the Archard law to model the evolution of wear.

Variational analysis of the models includes existence and uniqueness results of weak solutions as well as results of continuous dependence of the solution on the data and parameters. Links between different mechanical models are discussed; for example, elasticity as a limiting case of viscoelasticity, perfect plasticity as a limiting case of viscoplasticity, Signorini nonpenetration condition as a limiting case of the normal compliance contact condition. In carrying out the variational analysis we systematically use results on elliptic and evolutionary variational inequalities, convex analysis, nonlinear equations with monotone operators and fixed points of operators.

Two kinds of approximation schemes are introduced and analyzed. When only the spatial variables are discretized, we obtain semi-discrete schemes. If both the spatial and temporal variables are discretized, we arrive at fully discrete schemes. For both kinds of schemes we prove existence and uniqueness results. We show convergence of the discrete solutions under the basic solution regularity available from the well-posedness results of the variational problems. We also present optimal order error estimates under additional regularity assumption on the solution.

To demonstrate the performance of the numerical schemes, a number of numerical simulations are discussed. The test problems range from one to three dimensional geometries. The finite element method is used to discretize the spatial domain and finite differences are used for the time derivatives. We describe in the book numerical results in the study of some model contact problems for elastic, viscoelastic and viscoplastic materials.

The book is intended to be self-contained and accessible to a large number of readers. It is divided into four parts, as described in the following.

Part I is devoted to the basic notions and results which are fundamental to the development later in this book. We review here the background on functional analysis, function spaces, finite difference approximations and finite element method. Then we apply these results in the study of elliptic and evolutionary variational inequalities. The material presented in this part is self-contained. It does not need any knowledge in Contact Mechanics and could be aimed at graduate students and researchers interested in a general treatment of variational problems and their numerical approximations.

Part II presents preliminary material in Contact Mechanics. We summarize here basic notions and general principles of Mechanics of Continua. Then we introduce contact boundary conditions with or without friction as well as constitutive laws which are used in the rest of the book. We also present preliminary results on variational and numerical analysis in contact problems and we apply these results in the study of models involving elastic bodies. The material in Part II is aimed at those readers who are interested in the mechanical background of contact problems and mathematical theory of some contact problems in elasticity.

Parts III and IV represent the main parts of the book. They deal with the study of quasistatic problems for Kelvin-Voigt viscoelastic materials and rate-type viscoplastic materials, respectively. These parts are written based on our original research. We consider here a number of problems with various contact and frictional or frictionless boundary conditions, for which we provide variational analysis and numerical approximations. For some of the contact problems, we include results of numerical simulations to show the performance of the numerical schemes. These two parts of the book would interest mainly researchers for an in-depth knowledge of the mathematical theory of quasistatic contact problems.

The list of the references at the end of the book is by no means exhaustive. It only includes papers or books that were used for or are directly connected with the subjects treated in this book. Each part is concluded with a section entitled *Bibliographical Notes* that discusses references on the principal results treated, as well as information on important topics related to, but not included in, the body of the text.

Each of the four parts of the book is divided into several chapters. All the chapters are numbered consecutively. Mathematical relations (equalities or inequalities) are numbered by chapter and their order of occurrence. For example, (5.3) is the third numbered mathematical relation in Chapter 5. Definitions, examples, problems, theorems, lemmas, corollaries, propositions and remarks are numbered consecutively within each chapter. For example, in Chapter 10, Problem 10.1 is followed by Theorem 10.2.

The present book is the result of three years of cooperation between the two authors. In writing it we have drawn on the results of our joint collaboration with numerous colleagues and friends to whom we address our thanks. We express our gratitude to Professor Meir Shillor for our beneficial cooperation as well as for the interesting discussions on the models treated in the book. We particularly thank Dr. J.R. Fernández-García who provided the numerical simulations included in Parts III and IV of the book. We extend our thanks to Dr. M. Barboteu who realized the numerical results in Part II of the book. We especially thank Professor Shing-Tung Yau for his support and encouragement of our work.

The work of W.H. was supported by NSF/DARPA under Grant DMS-9874015, James Van Allen Natural Science Fellowship, Faculty Development Award at the University of Iowa, and K.C. Wong Education Foundation. Part of the book manuscript was prepared when W.H. was visiting the State Key Laboratory of Scientific and Engineering Computing, Chinese Academy of Sciences, during the summer of 2000 through the support of the K.C. Wong Education Foundation. He particularly thanks Professor Lie-heng Wang for the warm hospitality.

W.H.
Iowa City

M.S.
Perpignan

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List of Symbols

Sets

$\mathbb{N}$: the set of positive integers

$\mathbb{Z}_+$: the set of non-negative integers

$\mathbb{R}$: the real line

$\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$: extended real line

$\mathbb{R}_+$: the set of non-negative numbers

$\mathbb{R}^d$: the d-dimensional Euclidean space

$\mathbb{S}^d$: the space of second order symmetric tensors on $\mathbb{R}^d$

$\mathbb{Q}^d_+$: the set of orthogonal matrices of order d with determinant 1

S_d: the unit sphere in $\mathbb{R}^d$

Ω: an open, bounded, connected set in $\mathbb{R}^d$ with a Lipschitz boundary Γ

Γ: the boundary of the domain Ω, that is decomposed as $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$ with Γ_1, Γ_2 and Γ_3 relatively open with respect to Γ

Γ_1: the part of the boundary where displacement condition is specified; meas $(\Gamma_1) > 0$ is assumed throughout the book

Γ_2: the part of the boundary where traction condition is specified

Γ_3: the part of the boundary for contact

$\overline{I} = [0, T]$: time interval of interest

$I = (0, T)$

Operators

$\boldsymbol{\varepsilon}$: deformation operator, i.e. $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$, $\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$ (page 97)

Div: divergence operator, i.e. $\mathrm{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j})$ (page 100)

Π^h: finite element interpolation operator (page 52)

$\mathcal{P}_K$: projection operator onto a set K (page 16)

$\mathcal{P}^h$: finite element projection operator (pages 77, 181)

$\mathcal{P}_{Q^h}$: finite element projection on Q^h (page 154)

Function spaces

$L^p(\Omega)$: the Lebesgue space of p-integrable functions, with the usual modification if $p = \infty$ (page 28)

$C^m(\overline{\Omega})$: the space of functions whose derivatives up to and including order m are continuous up to the boundary Γ (page 26)

$C_0^\infty(\Omega)$: the space of infinitely differentiable functions with compact support in Ω (page 26)

$W^{k,p}(\Omega)$: the Sobolev space of functions whose weak derivatives of orders less than or equal to k are p-integrable on Ω (page 30)

$H^k(\Omega) \equiv W^{k,2}(\Omega)$ (page 30)

$W_0^{k,p}(\Omega)$: the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ (page 30)

$H_0^k(\Omega) \equiv W_0^{k,2}(\Omega)$ (page 30)

$H^{-1}(\Omega)$: the dual of $H_0^1(\Omega)$ (page 34)

$H^{\frac{1}{2}}(\Gamma)$: a Sobolev space on Γ, defined as the range of the trace operator on $H^1(\Omega)$ (page 36)

$H^{-\frac{1}{2}}(\Gamma)$: the dual of $H^{\frac{1}{2}}(\Gamma)$ (page 36)

$H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$ (page 143)

H'_Γ: dual of H_Γ (page 144)

$H = \{\, \boldsymbol{v} = (v_1, \dots, v_d)^T : v_i \in L^2(\Omega),\ 1 \le i \le d \,\} = L^2(\Omega)^d$, inner product $(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega u_i(\boldsymbol{x})\, v_i(\boldsymbol{x})\, dx$ (page 142)

$Q = \{\, \boldsymbol{\tau} = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega),\ 1 \le i, j \le d \,\} = L^2(\Omega)_s^{d \times d}$, inner product $(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \sigma_{ij}(\boldsymbol{x})\, \tau_{ij}(\boldsymbol{x})\, dx$ (page 142)

$Q_1 = \{\, \boldsymbol{\tau} \in Q : \mathrm{Div}\,\boldsymbol{\tau} \in H \,\}$, inner product $(\boldsymbol{\sigma}, \boldsymbol{\tau})_{Q_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q + (\mathrm{Div}\,\boldsymbol{\sigma}, \mathrm{Div}\,\boldsymbol{\tau})_H$ (page 145)

$V = \{\, \boldsymbol{v} \in H^1(\Omega)^d : \boldsymbol{v} = \boldsymbol{0}$ a.e. on $\Gamma_1 \,\}$, inner product $(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q$ (page 144)

$V_1 = \{ \boldsymbol{v} \in V : v_\nu = 0 \text{ a.e. on } \Gamma_3 \}$ with the inner product $(\boldsymbol{u}, \boldsymbol{v})_V$ (page 144)

$V_2 = \{ \boldsymbol{v} \in V : v_\nu \le 0 \text{ a.e. on } \Gamma_3 \}$ with the inner product $(\boldsymbol{u}, \boldsymbol{v})_V$ (page 144)

X: a Hilbert space or its subset with inner product $(\cdot, \cdot)_X$, or a Banach space or its subset with norm $\| \cdot \|_X$

$C^m(\overline{I}; X) = \{ v \in C(\overline{I}; X) : v^{(j)} \in C(\overline{I}; X),\ j = 1, \ldots, m \}$ (page 37)

$L^p(I; X) = \{ v : \overline{I} \to X \text{ measurable} : \|v\|_{L^p(I;X)} < \infty \}$ (page 38)

$W^{k,p}(I; X) = \{ v \in L^p(0, T; X) : \|v^{(j)}\|_{L^p(I;X)} < \infty\ \forall j \le k \}$ (page 39)

$H^k(I; X) \equiv W^{k,2}(I; X)$ (page 40)

Other symbols

d: a positive integer taking the values $1, 2, 3$ in applications

c: a generic positive constant

$r_+ = \max\{0, r\}$: positive part of r

$\boldsymbol{\nu}$: unit outward normal on the boundary of Ω

$\langle \cdot, \cdot \rangle$: the duality pairing between $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$ (page 36)

$\langle \cdot, \cdot \rangle_\Gamma$: the duality pairing between H_Γ and H'_Γ (page 144)

$\forall$: for any

$\exists$: there exist(s)

$\overline{A}$: closure of the set A

$\operatorname{int} A$ or $\overset{\circ}{A}$: interior of the set A

∂A: boundary of the set A

δ_{ij}: the Kronecker delta

s.t.: such that

a.e.: almost everywhere

iff: if and only if

$O(h)$: for some constant $c > 0$ independent of h such that $|O(h)| \le c\, h$

$\Delta w_n = w_n - w_{n-1}$: backward difference

$\delta_n w_n = \Delta w_n / k_n$: backward divided difference

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Phenomena of contact between deformable bodies or between deformable and rigid bodies abound in industry and in everyday life. A few simple examples are brake pads with wheels, tires on roads, and pistons with skirts. Common industrial processes such as metal forming and metal extrusion involve contact evolutions. Because of the importance of contact processes in structural and mechanical systems, considerable effort has been put into modeling and numerical simulations.

This book introduces readers to a mathematical theory of contact problems involving deformable bodies. It covers mechanical modeling, mathematical formulations, variational analysis, and the numerical solution of the associated formulations. The authors give a complete treatment of some contact problems by presenting arguments and results in modeling, analysis, and numerical simulations.

Variational analysis of the models includes existence and uniqueness results of weak solutions, as well as results of continuous dependence of the solution on the data and parameters. Also discussed are links between different mechanical models.

In carrying out the variational analysis, the authors systematically use results on elliptic and evolutionary variational inequalities, convex analysis, nonlinear equations with monotone operators, and fixed points of operators.

Prerequisites include basic functional analysis, variational formulations of partial differential equation problems, and numerical approximations. The text is suitable for graduate students and researchers in applied mathematics, computational mathematics, and computational mechanics.

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