Alexander Grigor’yan

Heat Kernel and Analysis on Manifolds
Heat Kernel and Analysis on Manifolds
Heat Kernel and Analysis on Manifolds

Alexander Grigor’yan
To my wife Tatiana
## Contents

Preface xi

Chapter 1. Laplace operator and the heat equation in $\mathbb{R}^n$ 1
  1.1. Historical background 1
  1.2. The Green formula 2
  1.3. The heat equation 4
  Notes 13

Chapter 2. Function spaces in $\mathbb{R}^n$ 15
  2.1. Spaces $C^k$ and $L^p$ 15
  2.2. Convolution and partition of unity 17
  2.3. Approximation of integrable functions by smooth ones 20
  2.4. Distributions 23
  2.5. Approximation of distributions by smooth functions 28
  2.6. Weak derivatives and Sobolev spaces 34
  2.7. Heat semigroup in $\mathbb{R}^n$ 40
  Notes 47

Chapter 3. Laplace operator on a Riemannian manifold 49
  3.1. Smooth manifolds 49
  3.2. Tangent vectors 53
  3.3. Riemannian metric 56
  3.4. Riemannian measure 59
  3.5. Divergence theorem 64
  3.6. Laplace operator and weighted manifolds 67
  3.7. Submanifolds 70
  3.8. Product manifolds 72
  3.9. Polar coordinates in $\mathbb{R}^n, S^n, H^n$ 74
  3.10. Model manifolds 80
  3.11. Length of paths and the geodesic distance 85
  3.12. Smooth mappings and isometries 91
  Notes 95

Chapter 4. Laplace operator and heat equation in $L^2(M)$ 97
  4.1. Distributions and Sobolev spaces 97
  4.2. Dirichlet Laplace operator and resolvent 103
  4.3. Heat semigroup and $L^2$-Cauchy problem 112
Chapter 5. Weak maximum principle and related topics
  5.1. Chain rule in $W^1_0$  
  5.2. Chain rule in $W^1$  
  5.3. Markovian properties of resolvent and the heat semigroup  
  5.4. Weak maximum principle  
  5.5. Resolvent and the heat semigroup in subsets  

Notes

Chapter 6. Regularity theory in $\mathbb{R}^n$
  6.1. Embedding theorems  
  6.2. Two technical lemmas  
  6.3. Local elliptic regularity  
  6.4. Local parabolic regularity  

Notes

Chapter 7. The heat kernel on a manifold
  7.1. Local regularity issues  
  7.2. Smoothness of the semigroup solutions  
  7.3. The heat kernel  
  7.4. Extension of the heat semigroup  
  7.5. Smoothness of the heat kernel in $t,x,y$  
  7.6. Notes  

Notes

Chapter 8. Positive solutions
  8.1. The minimality of the heat semigroup  
  8.2. Extension of resolvent  
  8.3. Strong maximum/minimum principle  
  8.4. Stochastic completeness  

Notes

Chapter 9. Heat kernel as a fundamental solution
  9.1. Fundamental solutions  
  9.2. Some examples  
  9.3. Eternal solutions  

Notes

Chapter 10. Spectral properties
  10.1. Spectra of operators in Hilbert spaces  
  10.2. Bottom of the spectrum  
  10.3. The bottom eigenfunction  
  10.4. The heat kernel in relatively compact regions  
  10.5. Minimax principle  
  10.6. Discrete spectrum and compact embedding theorem  
  10.7. Positivity of $\lambda_1$  
  10.8. Long time asymptotic of $\log p_t$  

Notes
<table>
<thead>
<tr>
<th>Chapter 11. Distance function and completeness</th>
<th>295</th>
</tr>
</thead>
<tbody>
<tr>
<td>11.1. The notion of completeness</td>
<td>295</td>
</tr>
<tr>
<td>11.2. Lipschitz functions</td>
<td>296</td>
</tr>
<tr>
<td>11.3. Essential self-adjointness</td>
<td>301</td>
</tr>
<tr>
<td>11.4. Stochastic completeness and the volume growth</td>
<td>303</td>
</tr>
<tr>
<td>11.5. Parabolic manifolds</td>
<td>313</td>
</tr>
<tr>
<td>11.6. Spectrum and the distance function</td>
<td>317</td>
</tr>
<tr>
<td>Notes</td>
<td>319</td>
</tr>
<tr>
<td>Chapter 12. Gaussian estimates in the integrated form</td>
<td>321</td>
</tr>
<tr>
<td>12.1. The integrated maximum principle</td>
<td>321</td>
</tr>
<tr>
<td>12.2. The Davies-Gaffney inequality</td>
<td>324</td>
</tr>
<tr>
<td>12.3. Upper bounds of higher eigenvalues</td>
<td>327</td>
</tr>
<tr>
<td>12.4. Semigroup solutions with a harmonic initial function</td>
<td>331</td>
</tr>
<tr>
<td>12.5. Takeda’s inequality</td>
<td>333</td>
</tr>
<tr>
<td>Notes</td>
<td>339</td>
</tr>
<tr>
<td>Chapter 13. Green function and Green operator</td>
<td>341</td>
</tr>
<tr>
<td>13.1. The Green operator</td>
<td>341</td>
</tr>
<tr>
<td>13.2. Superaveraging functions</td>
<td>348</td>
</tr>
<tr>
<td>13.3. Local Harnack inequality</td>
<td>351</td>
</tr>
<tr>
<td>13.4. Convergence of sequences of $\alpha$-harmonic functions</td>
<td>355</td>
</tr>
<tr>
<td>13.5. The positive spectrum</td>
<td>357</td>
</tr>
<tr>
<td>13.6. Green function as a fundamental solution</td>
<td>359</td>
</tr>
<tr>
<td>Notes</td>
<td>362</td>
</tr>
<tr>
<td>Chapter 14. Ultracontractive estimates and eigenvalues</td>
<td>365</td>
</tr>
<tr>
<td>14.1. Ultracontractivity and heat kernel bounds</td>
<td>365</td>
</tr>
<tr>
<td>14.2. Faber-Krahn inequalities</td>
<td>367</td>
</tr>
<tr>
<td>14.3. The Nash inequality</td>
<td>368</td>
</tr>
<tr>
<td>14.4. The function classes $\mathbb{L}$ and $\Gamma$</td>
<td>371</td>
</tr>
<tr>
<td>14.5. Faber-Krahn implies ultracontractivity</td>
<td>380</td>
</tr>
<tr>
<td>14.6. Ultracontractivity implies a Faber-Krahn inequality</td>
<td>381</td>
</tr>
<tr>
<td>14.7. Lower bounds of higher eigenvalues</td>
<td>384</td>
</tr>
<tr>
<td>14.8. Faber-Krahn inequality on direct products</td>
<td>386</td>
</tr>
<tr>
<td>Notes</td>
<td>388</td>
</tr>
<tr>
<td>Chapter 15. Pointwise Gaussian estimates I</td>
<td>391</td>
</tr>
<tr>
<td>15.1. $L^2$-mean value inequality</td>
<td>391</td>
</tr>
<tr>
<td>15.2. Faber-Krahn inequality in balls</td>
<td>397</td>
</tr>
<tr>
<td>15.3. The weighted $L^2$-norm of heat kernel</td>
<td>399</td>
</tr>
<tr>
<td>15.4. Faber-Krahn inequality in unions of balls</td>
<td>402</td>
</tr>
<tr>
<td>15.5. Off-diagonal upper bounds</td>
<td>404</td>
</tr>
<tr>
<td>15.6. Relative Faber-Krahn inequality and Li-Yau upper bounds</td>
<td>409</td>
</tr>
<tr>
<td>Notes</td>
<td>414</td>
</tr>
</tbody>
</table>
Preface

The development of Mathematics in the past few decades has witnessed an unprecedented rise in the usage of the notion of heat kernel in the diverse and seemingly remote sections of Mathematics. In the paper [217], titled “The ubiquitous heat kernel”, Jay Jorgenson and Serge Lang called the heat kernel “…a universal gadget which is a dominant factor practically everywhere in mathematics, also in physics, and has very simple and powerful properties.”

Already in a first Analysis course, one sees a special role of the exponential function \( t \mapsto e^{at} \). No wonder that a far reaching generalization of the exponential function – the heat semigroup \( \{e^{-tA}\}_{t\geq 0} \), where \( A \) is a positive definite linear operator, plays similarly an indispensable role in Mathematics and Physics, not the least because it solves the associated heat equation \( \dot{u} + Au = 0 \). If the operator \( A \) acts in a function space then frequently the action of the semigroup \( e^{-tA} \) is given by an integral operator, whose kernel is called then the heat kernel of \( A \).

 Needless to say that any knowledge of the heat kernel, for example, upper and/or lower estimates, can help in solving various problems related to the operator \( A \) and its spectrum, the solutions to the heat equation, as well as to the properties of the underlying space. If in addition the operator \( A \) is Markovian, that is, generates a Markov process (for example, this is the case when \( A \) is a second order elliptic differential operator), then one can use information about the heat kernel to answer questions concerning the process itself.

This book is devoted to the study of the heat equation and the heat kernel of the Laplace operator on Riemannian manifolds. Over 140 years ago, in 1867, Eugenio Beltrami [29] introduced the Laplace operator for a Riemannian metric, which is also referred to as the Laplace-Beltrami operator. The next key step towards analysis of this operator was made in 1954 by Matthew Gaffney [126], who showed that on geodesically complete manifolds the Laplace operator is essentially self-adjoint in \( L^2 \). Gaffney also proved in [127] the first non-trivial sufficient condition for the stochastic completeness of the heat semigroup, that is, for the preservation of the \( L^1 \)-norm by this semigroup. Nearly at the same time S. Minakshisundaram [275] constructed the heat kernel on compact Riemannian manifolds using the parametrix method.
However, it was not until the mid-1970s when the geometric analysis of the Laplace operator and the heat equation was revolutionized in the groundbreaking work of Shing-Tung Yau, which completely reshaped the area. The culmination of this work was the proof by Li and Yau [258] in 1986 of the parabolic Harnack inequality and the heat kernel two-sided estimates on complete manifolds of non-negative Ricci curvature, which stimulated further research on heat kernel estimates by many authors. Apart from the general wide influence on geometric analysis, the gradient estimates of Li and Yau motivated Richard Hamilton in his program on Ricci flow that eventually lead to the resolution of the Poincaré conjecture by Grigory Perel’mann, which can be viewed as a most spectacular application of heat kernels in geometry\(^1\).

Another direction in heat kernel research was developed by Brian Davies [96] and Nick Varopoulos [353], [355], who used primarily function-analytic methods to relate heat kernel estimates to certain functional inequalities.

The purpose of this book is to provide an accessible for graduate students introduction to the geometric analysis of the Laplace operator and the heat equation, which would bridge the gap between the foundations of the subject and the current research. The book focuses on the following aspects of these notions, which form separate chapters or groups of chapters.

I. Local geometric background. A detailed introduction to Riemannian geometry is given, with emphasis on construction of the Riemannian measure and the Riemannian Laplace operator as an elliptic differential operator of second order, whose coefficients are determined by the Riemannian metric tensor.

II. Spectral-theoretic properties. It is a crucial observation that the Laplace operator can be extended to a self-adjoint operator in \(L^2\) space, which enables one to invoke the spectral theory and functional calculus of self-adjoint operator and, hence, to construct the associated heat semigroup. To treat properly the domains of the self-adjoint Laplacian and that of the associated energy form, one needs the Sobolev function spaces on manifolds. A detailed introduction to the theory of distributions and Sobolev spaces is given in the setting of \(\mathbb{R}^n\) and Riemannian manifolds.

III. Markovian properties and maximum principles. The above spectral-theoretic aspect of the Laplace operator exploits its ellipticity and symmetry. The fact that its order is 2 leads to the so-called Markovian properties, that is, to maximum and minimum principles for solutions to the Laplace equation and the heat equation. Various versions of maximum/minimum principles are presented in different parts of the book, in the weak, normal, and strong forms. The Markovian properties are tightly related to the diffusion Markov process associated with the Laplacian, where is reflected in

---

\(^1\)Another striking application of heat kernels is the heat equation approach to the Atiyah-Singer index theorem – see [12], [132], [317].
the terminology. However, we do not treat stochastic processes here, leaving this topic for a prospective second volume.

IV. Smoothness properties. As it is well-known, elliptic and parabolic equations feature an added regularity phenomenon, when the degree of smoothness of solutions is higher than a priori necessary. A detailed account of the local regularity theory in $\mathbb{R}^n$ (and consequently on manifolds) is given for elliptic and parabolic operators with smooth coefficients. This includes the study of the smoothness of solutions in the scale of Sobolev spaces of positive and negative orders, as well as the embedding theorems of Sobolev spaces into $C^k$. The local estimates of solutions are used, in particular, to prove the existence of the heat kernel on an arbitrary manifold.

V. Global geometric aspects. These are those properties of solutions which depend on the geometry of the manifold in the large, such as the essential self-adjointness of the Laplace operator (that is, the uniqueness of the self-adjoint extension), the stochastic completeness of the heat kernel, the uniqueness in the bounded Cauchy problem for the heat equation, and the quantitative estimates of solutions, in particular, of the heat kernel. A special attention is given to upper bounds of the heat kernel, especially the on-diagonal upper bounds with the long-time dependence, and the Gaussian upper bounds reflecting the long-distance behavior. The lower bounds as well as the related uniform Harnack inequalities and gradient estimates are omitted and will be included in the second volume.

The prerequisites for reading of this book are Analysis in $\mathbb{R}^n$ and the basics of Functional Analysis, including Measure Theory, Hilbert spaces, and Spectral Theorem for self-adjoint operators (the necessary material from Functional Analysis is briefly surveyed in Appendix). The book can be used as a source for a number of graduate lecture courses on the following topics: Riemannian Geometry, Analysis on Manifolds, Sobolev Spaces, Partial Differential Equations, Heat Semigroups, Heat Kernel Estimates, and others. In fact, it grew up from a graduate course “Analysis on Manifolds” that was taught by the author in 1995-2005 at Imperial College London and in 2002, 2005 at Chinese University of Hong Kong.

The book is equipped with over 400 exercises whose level of difficulty ranges from “general nonsense” to quite involved. The exercises extend and illustrate the main text, some of them are used in the main text as lemmas. The detailed solutions of the exercises (about 200 pages) as well as their LATEX sources are available on the web page of the AMS

http://www.ams.org/bookpages/amsip-47

where also additional material on the subject of the book will be posted.

The book has little intersection with the existing monographs on the subject. The above mentioned upper bounds of heat kernels, which were obtained mostly by the author in 1990s, are presented for the first time in a book format. However, the background material is also significantly different from the previous accounts. The main distinctive feature of the foundation
part of this book is a new method of construction of the heat kernel on an arbitrary Riemannian manifold. Since the above mentioned work by Minakshisundaram, the traditional method of constructing the heat kernel was by using the parametrix method (see, for example, [36], [37], [51], [317], [326]). However, a recent development of analysis on metric spaces, including fractals (see [22], [186], [187], [224]), has lead to emergence of other methods that are not linked so much to the local Euclidean structure of the underlying space.

Although singular spaces are not treated here, we still employ whenever possible those methods that could be applied also on such spaces. This desire has resulted in the abandonment of the parametrix method as well as the tools using smooth hypersurfaces such as the coarea formula and the boundary regularity of solutions, sometimes at expense of more technical arguments. Consequently, many proofs in this book are entirely new, even for the old well-known properties of the heat kernel and the Green function. A number of key theorems are presented with more than one proof, which should provide enough flexibility for building lecture courses for audiences with diverse background.

The material of Chapters 1-10, the first part of Chapter 11, and Chapter 13, belongs to the foundation of the subject. The rest of the book – the second part of Chapter 11, Chapters 12 and 14-16, contains more advanced results, obtained in the 1980s-1990s.

Let us briefly describe the contents of the individual chapters.

Chapters 1, 2, 6 contain the necessary material on the analysis in $\mathbb{R}^n$ and the regularity theory of elliptic and parabolic equations in $\mathbb{R}^n$. They do not depend on the other chapters and can be either read independently or used as a reference source on the subject.

Chapter 3 contains a rather elementary introduction to Riemannian geometry, which focuses on the Laplace-Beltrami operator and the Green formula.

Chapter 4 introduces the Dirichlet Laplace operator as a self-adjoint operator in $L^2$, which allows then to define the associated heat semigroup and to prove its basic properties. The spectral theorem is the main tool in this part.

Chapter 5 treats the Markovian properties of the heat semigroup, which amounts to the chain rule for the weak gradient, and the weak maximum principle for elliptic and parabolic problems. The account here does not use the smoothness of solutions; hence, the main tools are the Sobolev spaces.

Chapter 7 introduces the heat kernel on an arbitrary manifold as the integral kernel of the heat semigroup. The main tool is the regularity theory of Chapter 6, transplanted to manifolds. The existence of the heat kernel is derived from a local $L^2 \to L^\infty$ estimate of the heat semigroup, which in turn is a consequence of the Sobolev embedding theorem and the regularity theory. The latter implies also the smoothness of the heat kernel.
Chapter 8 deals with a number of issues related to the positivity or boundedness of solutions to the heat equation, which can be regarded as an extension of Chapter 5 using the smoothness of the solutions. It contains the results on the minimality of the heat semigroup and resolvent, the strong minimum principle for positive supersolutions, and some basic criteria for the stochastic completeness.

Chapter 9 treats the heat kernel as a fundamental solution. Based on that, some useful tools are introduced for verifying that a given function is the heat kernel, and some examples of heat kernels are given.

Chapter 10 deals with basic spectral properties of the Dirichlet Laplacian. It contains the variational principle for the bottom of the spectrum $\lambda_1$, the positivity of the bottom eigenfunction, the discreteness of the spectrum and the positivity of $\lambda_1$ in relatively compact domains, and the characterization of the long time behavior of the heat kernel in terms of $\lambda_1$.

Chapter 11 contains the material related to the use of the geodesic distance. It starts with the properties of Lipschitz functions, in particular, their weak differentiability, which allows then to use Lipschitz functions as test functions in various proofs. The following results are proved using the distance function: the essential self-adjointness of the Dirichlet Laplacian on geodesically complete manifolds, the volume tests for the stochastic completeness and parabolicity, and the estimates of the bottom of the spectrum.

Chapter 12 is the first of the four chapters dealing with upper bounds of the heat kernel. It contains the results on the integrated Gaussian estimates that are valid on an arbitrary manifold: the integrated maximum principle, the Davies-Gaffney inequality, Takeda’s inequality, and some consequences. The proofs use the carefully chosen test functions based on the geodesic distance.

Chapter 13 is devoted to the Green function of the Laplace operator, which is constructed by integrating the heat kernel in time. Using the Green function together with the strong minimum principle allows to prove the local Harnack inequality for $\alpha$-harmonic functions and its consequences – convergence theorems. As an example of application, the existence of the ground state on an arbitrary manifold is proved. Logically this Chapter belongs to the foundations of the subject and should have been placed much earlier in the sequence of the chapters. However, the proof of the local Harnack inequality requires one of the results of Chapter 12, which has necessitated the present order.

Chapter 14 deals with the on-diagonal upper bounds of the heat kernel, which requires additional hypothesis on the manifold in question. Normally such hypotheses are stated in terms of some isoperimetric or functional inequalities. We use here the approach based on the Faber-Krahn inequality for the bottom eigenvalue, which creates useful links with the spectral properties. The main result is that, to a certain extent, the on-diagonal upper bounds of the heat kernel are equivalent to the Faber-Krahn inequalities.
Chapter 15 continues the topic of the Gaussian estimates. The main technical result is Moser’s mean-value inequality for solutions of the heat equation, which together with the integrated maximum principle allows to obtain pointwise Gaussian upper bounds of the heat kernel. We consider such estimates in the following three settings: arbitrary manifolds, the manifolds with the global Faber-Krahn inequality, and the manifolds with the relative Faber-Krahn inequality that leads to the Li-Yau estimates of the heat kernel.

Chapter 16 introduces alternative tools to deal with the Gaussian estimates. The main point is that the Gaussian upper bounds can be deduced directly from the on-diagonal upper bounds, although in a quite elaborate manner. As an application of these techniques, some on-diagonal lower estimates are proved.

Finally, Appendix A contains some reference material as was already mentioned above.

ACKNOWLEDGMENTS. The book was typeset in \LaTeX using an excellent editor Scientific Workplace by TCI Software Research and MacKichan Software.

In the process of writing this book I was affiliated (permanently or temporarily) with the following institutions: Imperial College London, Institute of Henry Poincaré Paris, Chinese University of Hong Kong, Research Institute of Mathematical Sciences Kyoto, Institute of Control Sciences Moscow, University of Bielefeld, and ETH Zurich, with the support of the appropriate funding bodies.

However, the major part of the book was written during my three stays, totalling to twelve months, at the Institute of Mathematical Sciences of the Chinese University of Hong Kong, and I am very grateful to Professor Shing-Tung Yau for giving me that excellent opportunity. His support and encouragement have been paramount to me at all stages of my work.

Writing about maximum principle brings up memories of my teacher Eugene Landis. His masterful use of maximum principles has never been surpassed. My entire education in Analysis was hugely influenced by Landis, which has left an imprint on the style and choice of the material for this book.

A special thank is due to the late Serge Lang for useful discussions of the structure of the book.

It is a great pleasure to thank those colleagues of mine who have fruitfully affected my work in various ways: Martin Barlow, Alexander Bendikov, Isaac Chavel, Thierry Coulhon, Józef Dodziuk, Brian Davies, Wolfhard Hansen, Elton Pei Hsu, Jiaxin Hu, Vladimir Kondratiev, Takashi Kumagai, Ka-Sing Lau, Peter Li, Terry Lyons, Vladimir Maz’ya, Minoru Murata, Nikolai Nadirashvili, Michael Röckner, Laurent Saloff-Coste, Theo Sturm, Nina Ural’tseva.
Last but not least I am indebted to my family and especially to my wife Tatiana for inspiration and support.

Alexander Grigor’yan
London - Paris - Hong Kong - Kyoto - Moscow - Bielefeld - Zurich
2002-2009
Bibliography


303. Pittet Ch., Saloff-Coste L., A survey on the relationship between volume growth, isoperimetry, and the behavior of simple random walk on Cayley graphs, with examples, preprint
325. Saloff-Coste L., Pseudo-Poincaré inequalities and applications to Sobolev inequalities, preprint
358. Woess W., Random walks on infinite graphs and groups - a survey on selected topics, Bull. LMS, 26 (1994) 1-60.
370. Zhang Q.S., Large time behavior of Schrodinger heat kernels and applications, 

Some notation

- \( \mathbb{R}_+ \equiv (0, +\infty) \)
- \( \text{esup} \) – the essential supremum
- \( \text{einf} \) – the essential infimum
- \( f_+ \equiv \frac{1}{2} (|f| + f) \) - the positive part
- \( f_- \equiv \frac{1}{2} (|f| - f) \) - the negative part
- \( [f]_a^b \equiv f(b) - f(a) \)
- \( \log_+ x \equiv (\log x)^+ \)
- \( \simeq \) “comparable to”; namely, \( f(x) \simeq g(x) \) if there exists a constant \( C > 0 \) such that \( C^{-1} g(x) \leq f(x) \leq C g(x) \) for all \( x \) from a specified domain.
- \( B_r(x) \) – a ball in \( \mathbb{R}^n \), that is, \( B_r(x) = \{ y \in \mathbb{R}^n : |x - y| < r \} \)
- \( B_r \equiv B_r(0) = \{ y \in \mathbb{R}^n : |y| < r \} \)
- \( \omega_n \) – the area of the unit \((n-1)\)-sphere in \( \mathbb{R}^n \).
- \( 1_A \) – the indicator function of a set \( A \), that is, \( 1_A(x) = 1 \) if \( x \in A \) and \( 1_A(x) = 0 \) otherwise.
- \( \subset \) “compact inclusion”; \( A \subset B \) means that the closure \( \overline{A} \) of the set \( A \) is compact and \( \overline{A} \subset B \).
- \( \mathcal{H} \) – a Hilbert space
- \( \rightharpoonup \) the sign of the weak convergence (in a Hilbert space)
- \( M \) – a Riemannian manifold
- \( x \to \infty \) – a sequence of points on a manifold eventually leaving any compact set.
- \( g \) – the Riemannian metric on \( M \)
- \( \mu \) – a reference measure on \( M \)
- \( \Delta \mu \) – the weighted Laplace operator on \( M \)
- \( \mathcal{L} \) – the Dirichlet Laplace operator on \( M \)
- \( L^p(M, \mu) \) – the Lebesgue function space
- \( \| \cdot \|_p \equiv \| \cdot \|_{L^p} \)
- \( u \leq v \mod W^1_0 \) means that \( u \leq v + w \) where \( w \in W^1_0 \).
- \( d \) – the geodesic distance on \( M \)
- \( B(x,r) \) – a geodesic ball on \( M \) with respect to the geodesic distance \( d(x,y) \).
Conventions.

- Summation is assumed over repeated indices. For example,

\[ \xi_i x^i = \sum_{i=1}^{n} \xi_i x^i, \quad a^{ij} u_j = \sum_{j=1}^{n} a^{ij} u_j, \quad g^{ij} v^i v^j = \sum_{i,j=1}^{n} g^{ij} v^i v^j, \]

etc.

- Letters \( c, C, c', C' \), etc denote positive constants (depending on specified parameters) whose value may change at each occurrence.

- positive ≡ strictly positive, negative ≡ strictly negative, decreasing ≡ non-increasing, increasing ≡ non-decreasing
Index

|α| - order of multiindex, 15

[α] - the weighted order, 170

\(B(M)\) - the class of Borel measurable functions on \(M\), 59

\(B(x, r)\) - the geodesic ball, 89

\(B_r(x)\) - the Euclidean ball, 17

\(C(\Omega)\), 15

\(C_b(M)\), 202

\(C_b(\Omega)\), 120

\(C_b(\mathbb{R}^n)\), 40

\(C^k(M)\), 51

\(C^k(\Omega)\), 15

\(C^{k_0}(M)\), 51

\(C^{k_0}(\mathbb{R}^n)\), 41

\(C^\infty(\Omega)\), 16

\(C^\infty(\Omega)\), 16

\(\hookrightarrow\), 16

\(\ll\), 16, 49

\(\simeq\), 36, 93

\(\subset\subset\), 186

\(\mathcal{D}\), 24, 97

\(\mathcal{D}'\), 24, 97

\(\mathcal{W}^\infty_{\text{loc}}\), 186

\(\mathcal{D}'(M)\), 97

\(\mathcal{D}'(\Omega)\), 24

\(\mathcal{D}(M)\), 97

\(\mathcal{D}(\Omega)\), 23

\(\mathcal{D}'(M)\), 98

\(\mathcal{D}(M)\), 98

\(\delta_{ij}\) - the Kronecker delta, 56

\(\Delta\) - the Laplace operator, 1

\(\Delta_{\mu}\) - the weighted Laplace operator, 68

\(\text{diag}\), 260

\(d(x, y)\) - the geodesic distance, 86

\(dJ\) - the tangent map, 92

\(E_D(t, x)\), 399

\(E_U\) - the spectral measure of \(U\), 266

\(E_\lambda\) - a spectral resolution, 111, 113, 132, 190, 266, 449, 452

\(f \ast g\) - convolution, 7, 17

\(\varphi_{\varepsilon}\) - mollifier, 18

\(G\) - the Green operator, 341

\(g(x, y)\) - the Green function, 342

\(g^\Omega(x, y)\) - the Green function in \(\Omega\), 342

\(g\) - a Riemannian metric, 56

\(g^{-1}\), 58

\(g_{\mathbb{R}^n}\) - the canonical metric on \(\mathbb{R}^n\), 77

\(g_{\mathbb{H}^n}\) - the canonical metric on \(\mathbb{H}^n\), 57

\(g_{\mathbb{S}^n}\) - the canonical metric on \(\mathbb{S}^n\), 72

\(\Gamma\) - a function class, 372

\(\Gamma_{\delta}\) - a function class, 376

\(\Gamma_{\delta}\) - a function class, 376

\(Gf\), 341

\(G^{\delta}f\), 341

\(g_{ij}\), 57

\(g^{ij}\), 58

\(\langle \cdot, \cdot \rangle\) - inner product of tangent vectors, 57

\(\langle \cdot, \cdot \rangle\) - pairing of vectors and covectors, 56

\(\langle f, g \rangle_{L^2}\) - the inner product in \(L^2\), 440

\(J_*\) - pullback operator, 92

\(\mathcal{L}\) - the Dirichlet Laplace operator, 105

\(\mathcal{L}^\Omega\) - the Dirichlet Laplace operator in \(\Omega\), 144

\(\Lambda\) - a Faber-Krahn function, 367

\(\Lambda(M)\) - the class of Lebesgue measurable functions on \(M\), 59

\(\lambda_k(\Omega)\), 277

\(\lambda_{\min}(A)\), 265

\(\lambda_{\min}(M)\), 111, 271

\(\mathcal{L}\) - a function class, 371

\(\mathcal{L}_{\delta}\) - a function class, 376

\(\mathcal{L}_{\delta}\) - a function class, 376

\(\ell(\gamma)\) - the length of a path, 86

\(\text{Lip}(M)\), 296
**INDEX**

$Lip_0 (M)$, 299
$Lip_{loc} (M)$, 299
$log_+$, 378
$L^p (M)$, 98, 440
$L^p_{loc} (M)$, 98
$L^p (\Omega)$, 16
$L^p (M)$, 98
$L^p_{loc} (M)$, 99
$m (U)$, 267
$\nabla$, 6, 43, 58
$\parallel \cdot \parallel_{C^k}$, 16
$\parallel \cdot \parallel_{L^p}$, 440
$\parallel \cdot \parallel_p$, 439
$\parallel \cdot \parallel_{V^k}$, $k < 0$, 171
$\parallel \cdot \parallel_{V^k}$, $k \geq 0$, 170
$\parallel \cdot \parallel_{W^k}$, $k < 0$, 37
$\parallel \cdot \parallel_{W^k}$, $k \geq 0$, 34
$\parallel \cdot \parallel_{V^k_{\text{loc}}}$, 183
$\parallel \cdot \parallel_{p \rightarrow q}$, 365
$\parallel \cdot \parallel_{Lip}$, 296
$P_t$ - the heat semigroup
$\quad = e^{-t\mathcal{L}}$, 115, 117, 130
$\quad$ a convolution operator, 40
$\quad$ a smooth version of $e^{-t\mathcal{L}}$, 191
$\quad$ an integral operator, 201
$P_t^\Omega$ - the heat semigroup in $\Omega$, 144
$p_t (x)$ - the heat kernel in $\mathbb{R}^n$, 4
$p_t (x, y)$ - the heat kernel, 198
$p_t (x, y)$, 191
$\mathcal{R} (f)$ - the Rayleigh quotient, 272
$R_\alpha$ - the resolvent, 106, 130, 219
$R_\alpha^\Omega$ - the resolvent in $\Omega$, 144
$r_\alpha (x, y)$ - the resolvent kernel, 262
$R^k$ - the iterated resolvent, 133
$\quad$ supp - support
$\quad$ of a continuous function, 3, 51
$\quad$ of a distribution, 26, 97
$\quad$ of a function from $L^1_{\text{loc}}$, 98
$u = w \mod W_0^1 (M)$, 135
$u \leq w \mod W_0^1 (M)$, 135
$V (x, r)$, 303, 409
$V^k (\Omega)$, $k < 0$, 171
$V^k (\Omega)$, $k \geq 0$, 170
$V^k_{\text{loc}} (\Omega)$, 171
$W^1 (M)$, 100
$W_0^1 (M)$, 104
$W_0^k (M)$, 127
$W^1_{\text{loc}} (M)$, 128
$W^2 (M)$, 104
$W_0^2 (M)$, 104
$W^2_{\text{loc}} (M)$, 130
$W^k (\Omega)$, $k < 0$, 37
$W^k (\Omega)$, $k \geq 0$, 34
$W_0^k (\Omega)$, 36, 158
$W^\infty (\Omega)$, 152
$W^\infty_{\text{loc}} (\Omega)$, 152
$W^k_{\text{loc}} (\Omega)$, $k < 0$, 38
$W^k_{\text{loc}} (\Omega)$, $k \geq 0$, 34
$W_0^k (M)$, 188
$W^{2k} (M)$, 183
$W^k_{\text{loc}} (M)$, 183
$W^\infty_{\text{loc}} (M)$, 186
$\omega_n$, 3, 82, 83

$\sigma$-Algebra, 435
Almost everywhere, 438
Anisotropic Sobolev spaces, 170
Area function, 82
Aronson, Donald G., 215, 339, 414
Atlas, 50
Azencott, Robert, 320

Basis in a Hilbert space, 432
Beltrami, Eugenio, ix
Bessel semigroup, 133
Bessel's inequality, 432
Borel set
$\quad$ in $\mathbb{R}^n$, 436
$\quad$ on a manifold, 59
Bottom eigenfunction, 275
Bottom of the spectrum, 265
Boukricha, Abderrahman, 362
Bounded convergence theorem, 114, 439
Bounded geometry, 312
Brooks, Robert, 320

$C$-manifold, 49
Canonical Euclidean metric, 57
Canonical hyperbolic metric, 77
Canonical spherical metric, 72
Carathéodory extension theorem, 435
Carlen, Eric A., 388
Carron, Gilles, 388
Cartan-Hadamard manifold, 368, 383
Cauchy problem, 4
$\quad$ in $L^2 (\mathbb{R}^n)$, 45
$L^2$-Cauchy problem, 112
Cauchy semigroup, 134
Cauchy-Schwarz inequality, 431, 440
Chain rule
$\quad$ for Lipschitz functions, 301
$\quad$ for strong derivatives, 121
$\quad$ for the Riemannian gradient, 59
<table>
<thead>
<tr>
<th>Term</th>
<th>Page Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>for the weighted Laplacian</td>
<td>69</td>
</tr>
<tr>
<td>in $W^1$, 128</td>
<td></td>
</tr>
<tr>
<td>in $W^1_0$, 123, 124</td>
<td></td>
</tr>
<tr>
<td>Chart</td>
<td>49</td>
</tr>
<tr>
<td>Chavel, Isaac</td>
<td>388</td>
</tr>
<tr>
<td>Cheeger’s inequality</td>
<td>275</td>
</tr>
<tr>
<td>$C^k$-norm</td>
<td>16</td>
</tr>
<tr>
<td>Closed operator</td>
<td>109, 446</td>
</tr>
<tr>
<td>Compact embedding theorem</td>
<td>214, 289</td>
</tr>
<tr>
<td>in $\mathbb{R}^n$, 158</td>
<td></td>
</tr>
<tr>
<td>Compact inclusion</td>
<td>16, 49</td>
</tr>
<tr>
<td>Compact operator</td>
<td>168, 434</td>
</tr>
<tr>
<td>Comparison principle</td>
<td>137</td>
</tr>
<tr>
<td>Complete measure</td>
<td>435</td>
</tr>
<tr>
<td>Completeness of $L^p$</td>
<td>440</td>
</tr>
<tr>
<td>Components</td>
<td></td>
</tr>
<tr>
<td>of a vector, 55</td>
<td></td>
</tr>
<tr>
<td>of the metric tensor, 57</td>
<td></td>
</tr>
<tr>
<td>Convergence</td>
<td></td>
</tr>
<tr>
<td>in $\mathcal{D}(\Omega)$, 23</td>
<td></td>
</tr>
<tr>
<td>in $\mathcal{D}(M)$, 97</td>
<td></td>
</tr>
<tr>
<td>Convex function</td>
<td>42</td>
</tr>
<tr>
<td>Convexity lemma</td>
<td>43</td>
</tr>
<tr>
<td>Convolution</td>
<td>17</td>
</tr>
<tr>
<td>Cotangent space</td>
<td>56</td>
</tr>
<tr>
<td>Coulhon, Thierry</td>
<td>388</td>
</tr>
<tr>
<td>Countable base</td>
<td>49</td>
</tr>
<tr>
<td>Counting measure</td>
<td>267</td>
</tr>
<tr>
<td>Covector</td>
<td>56</td>
</tr>
<tr>
<td>Cutoff function</td>
<td>19</td>
</tr>
<tr>
<td>Lipschitz, 300</td>
<td></td>
</tr>
<tr>
<td>on a manifold, 52</td>
<td></td>
</tr>
<tr>
<td>Davies, Edward Brian</td>
<td>x, 339</td>
</tr>
<tr>
<td>Davies-Gaffney inequality</td>
<td>326</td>
</tr>
<tr>
<td>De Broglie, Louis</td>
<td>2</td>
</tr>
<tr>
<td>De Giorgi, Ennio</td>
<td>181, 215, 414</td>
</tr>
<tr>
<td>Delta function</td>
<td>24</td>
</tr>
<tr>
<td>Density function</td>
<td>67</td>
</tr>
<tr>
<td>Density of measure</td>
<td>438</td>
</tr>
<tr>
<td>Diffeomorphism</td>
<td>92</td>
</tr>
<tr>
<td>Differential</td>
<td>56</td>
</tr>
<tr>
<td>Dirac, Paul</td>
<td>2</td>
</tr>
<tr>
<td>Dirichlet Laplace operator</td>
<td>105</td>
</tr>
<tr>
<td>Dirichlet problem</td>
<td>105</td>
</tr>
<tr>
<td>weak, 105, 111, 135</td>
<td></td>
</tr>
<tr>
<td>Discrete spectrum</td>
<td>265</td>
</tr>
<tr>
<td>Distribution</td>
<td></td>
</tr>
<tr>
<td>definition, 24</td>
<td></td>
</tr>
<tr>
<td>derivatives, 25</td>
<td></td>
</tr>
<tr>
<td>multiplication by a function, 25</td>
<td></td>
</tr>
<tr>
<td>non-negative, 136</td>
<td></td>
</tr>
<tr>
<td>on a manifold, 97</td>
<td></td>
</tr>
<tr>
<td>support, 26, 97</td>
<td></td>
</tr>
<tr>
<td>Distributional gradient</td>
<td>99</td>
</tr>
<tr>
<td>Distributional vector field</td>
<td>98</td>
</tr>
<tr>
<td>Divergence</td>
<td></td>
</tr>
<tr>
<td>on a manifold, 64</td>
<td></td>
</tr>
<tr>
<td>weighted, 68</td>
<td></td>
</tr>
<tr>
<td>Divergence theorem</td>
<td></td>
</tr>
<tr>
<td>in $\mathbb{R}^n$, 3</td>
<td></td>
</tr>
<tr>
<td>on a manifold, 64</td>
<td></td>
</tr>
<tr>
<td>Dodziuk, Józef</td>
<td>263, 429</td>
</tr>
<tr>
<td>Dominated convergence theorem</td>
<td>439, 441</td>
</tr>
<tr>
<td>Doob, 252</td>
<td></td>
</tr>
<tr>
<td>Doubling volume property</td>
<td>410</td>
</tr>
<tr>
<td>Eigenvalue</td>
<td>434</td>
</tr>
<tr>
<td>Eigenvector</td>
<td>434</td>
</tr>
<tr>
<td>Einstein, Albert</td>
<td>2</td>
</tr>
<tr>
<td>Elliptic operator</td>
<td>4, 162</td>
</tr>
<tr>
<td>Ellipticity constant</td>
<td>162</td>
</tr>
<tr>
<td>Embedding</td>
<td></td>
</tr>
<tr>
<td>of linear topological spaces, 16</td>
<td></td>
</tr>
<tr>
<td>Essential spectrum</td>
<td>265</td>
</tr>
<tr>
<td>Exhaustion sequence</td>
<td>52, 144</td>
</tr>
<tr>
<td>compact, 52, 201</td>
<td></td>
</tr>
<tr>
<td>Faber-Krahn inequality</td>
<td>367</td>
</tr>
<tr>
<td>in balls, 397</td>
<td></td>
</tr>
<tr>
<td>in unions of balls, 402</td>
<td></td>
</tr>
<tr>
<td>on direct products, 386</td>
<td></td>
</tr>
<tr>
<td>relative, 409</td>
<td></td>
</tr>
<tr>
<td>Faber-Krahn theorem</td>
<td>367</td>
</tr>
<tr>
<td>Fatou’s lemma</td>
<td>438</td>
</tr>
<tr>
<td>Fourier series</td>
<td>432</td>
</tr>
<tr>
<td>Fourier transform</td>
<td>8</td>
</tr>
<tr>
<td>inversion formula, 155</td>
<td></td>
</tr>
<tr>
<td>Fourier, Jean Baptiste Joseph</td>
<td>1</td>
</tr>
<tr>
<td>Friedrichs lemma</td>
<td>160</td>
</tr>
<tr>
<td>Friedrichs, Kurt Otto</td>
<td>181</td>
</tr>
<tr>
<td>Friedrichs-Poincaré inequality</td>
<td>159</td>
</tr>
<tr>
<td>Fubini’s theorem</td>
<td>442</td>
</tr>
<tr>
<td>Functional calculus of operators</td>
<td>453</td>
</tr>
<tr>
<td>Fundamental solution</td>
<td></td>
</tr>
<tr>
<td>of the heat equation, 243</td>
<td></td>
</tr>
<tr>
<td>of the Laplace operator</td>
<td>342, 359</td>
</tr>
<tr>
<td>regular, 243</td>
<td></td>
</tr>
<tr>
<td>Fundamental theorem of calculus</td>
<td>120</td>
</tr>
<tr>
<td>$\Gamma$-transform</td>
<td>372</td>
</tr>
<tr>
<td>Gaffney, Matthew P., ix</td>
<td>319</td>
</tr>
<tr>
<td>Gamma function</td>
<td>455</td>
</tr>
<tr>
<td>Gâteaux derivative</td>
<td>210</td>
</tr>
<tr>
<td>Gaussian upper bounds</td>
<td>391</td>
</tr>
<tr>
<td>Geodesic ball</td>
<td>89</td>
</tr>
<tr>
<td>Term</td>
<td>Page(s)</td>
</tr>
<tr>
<td>-------------------------------------------</td>
<td>-----------------</td>
</tr>
<tr>
<td>Geodesic completeness</td>
<td>295</td>
</tr>
<tr>
<td>Geodesic distance</td>
<td>86</td>
</tr>
<tr>
<td>Geodesics</td>
<td>86, 295</td>
</tr>
<tr>
<td>Gradient</td>
<td>58</td>
</tr>
<tr>
<td>Green formula</td>
<td>104</td>
</tr>
<tr>
<td>Green function</td>
<td>342</td>
</tr>
<tr>
<td>Green operator</td>
<td>341</td>
</tr>
<tr>
<td>Gross, Leonard</td>
<td>388</td>
</tr>
<tr>
<td>Ground state</td>
<td>358</td>
</tr>
<tr>
<td>Gushchin Anatolii Konstantinovich</td>
<td>320</td>
</tr>
<tr>
<td>$h$-transform</td>
<td>252</td>
</tr>
<tr>
<td>Hamilton, Richard</td>
<td>x</td>
</tr>
<tr>
<td>Hansen, Wolfhard</td>
<td>362</td>
</tr>
<tr>
<td>Hardy inequality</td>
<td>259</td>
</tr>
<tr>
<td>Harmonic function</td>
<td>83, 189, 229</td>
</tr>
<tr>
<td>$\alpha$-Harmonic function</td>
<td>229, 354, 356</td>
</tr>
<tr>
<td>Harnack inequality</td>
<td></td>
</tr>
<tr>
<td>in $\mathbb{R}^n$, 355</td>
<td></td>
</tr>
<tr>
<td>local, 353</td>
<td></td>
</tr>
<tr>
<td>Harnack principle</td>
<td>356</td>
</tr>
<tr>
<td>Hausdorff space</td>
<td>49</td>
</tr>
<tr>
<td>Heat kernel</td>
<td></td>
</tr>
<tr>
<td>asymptotics as $t \to \infty$, 292</td>
<td></td>
</tr>
<tr>
<td>existence, 191, 428</td>
<td></td>
</tr>
<tr>
<td>in half-space, 258</td>
<td></td>
</tr>
<tr>
<td>in $\mathbb{H}^n$, 256</td>
<td></td>
</tr>
<tr>
<td>in $\mathbb{R}^n$, 4</td>
<td></td>
</tr>
<tr>
<td>in Weyl's chamber, 258</td>
<td></td>
</tr>
<tr>
<td>integrated upper bound, 399, 422</td>
<td></td>
</tr>
<tr>
<td>Li-Yau upper estimate, 413</td>
<td></td>
</tr>
<tr>
<td>of a weighted manifold, 198</td>
<td></td>
</tr>
<tr>
<td>off-diagonal upper bound, 404, 410</td>
<td></td>
</tr>
<tr>
<td>on model manifolds, 251</td>
<td></td>
</tr>
<tr>
<td>on products, 249</td>
<td></td>
</tr>
<tr>
<td>on-diagonal lower bound, 424</td>
<td></td>
</tr>
<tr>
<td>on-diagonal upper bound, 380</td>
<td></td>
</tr>
<tr>
<td>smoothness, 198, 208</td>
<td></td>
</tr>
<tr>
<td>under change of measure, 252</td>
<td></td>
</tr>
<tr>
<td>under isometry, 250</td>
<td></td>
</tr>
<tr>
<td>Heat semigroup</td>
<td></td>
</tr>
<tr>
<td>in $\mathbb{R}^n$, 40</td>
<td></td>
</tr>
<tr>
<td>on a manifold, 115</td>
<td></td>
</tr>
<tr>
<td>Hermite polynomials</td>
<td>69</td>
</tr>
<tr>
<td>Hilbert space</td>
<td>431</td>
</tr>
<tr>
<td>Hilbert-Schmidt theorem</td>
<td>434</td>
</tr>
<tr>
<td>Hölder conjugate</td>
<td>439</td>
</tr>
<tr>
<td>Hölder inequality</td>
<td>439</td>
</tr>
<tr>
<td>Hopf-Rinow Theorem</td>
<td>295, 296</td>
</tr>
<tr>
<td>Hörmander, Lars Valter</td>
<td>181</td>
</tr>
<tr>
<td>Hyperbolic space</td>
<td>77</td>
</tr>
<tr>
<td>Induced measure</td>
<td>71</td>
</tr>
<tr>
<td>Induced metric</td>
<td>71</td>
</tr>
<tr>
<td>Infinity point $\infty$ on a manifold</td>
<td>141</td>
</tr>
<tr>
<td>Initial value problem</td>
<td>4</td>
</tr>
<tr>
<td>Integrable function</td>
<td>437</td>
</tr>
<tr>
<td>Integral maximum principle</td>
<td>321</td>
</tr>
<tr>
<td>Integration by parts formula</td>
<td>3</td>
</tr>
<tr>
<td>Isometric manifolds</td>
<td>92</td>
</tr>
<tr>
<td>Jacobian matrix</td>
<td>60</td>
</tr>
<tr>
<td>Jorgenson, Jay</td>
<td>ix</td>
</tr>
<tr>
<td>Khas’minskii, Rafail Zalmanovich</td>
<td>241</td>
</tr>
<tr>
<td>Krylov, Nikolai Vladimirovitch</td>
<td>181</td>
</tr>
<tr>
<td>Kusuoka, Shigeo</td>
<td>388</td>
</tr>
<tr>
<td>$L$-transform</td>
<td>372</td>
</tr>
<tr>
<td>Landis, Evgeniy Mikhailovich</td>
<td>182</td>
</tr>
<tr>
<td>Lang, Serge</td>
<td>ix</td>
</tr>
<tr>
<td>Laplace equation</td>
<td>1</td>
</tr>
<tr>
<td>Laplace operator</td>
<td></td>
</tr>
<tr>
<td>Dirichlet, 105</td>
<td></td>
</tr>
<tr>
<td>distributional, 99</td>
<td></td>
</tr>
<tr>
<td>in $\mathbb{R}^n$, 1</td>
<td></td>
</tr>
<tr>
<td>on a manifold, 67</td>
<td></td>
</tr>
<tr>
<td>weak, 99</td>
<td></td>
</tr>
<tr>
<td>weighted, 68</td>
<td></td>
</tr>
<tr>
<td>Laplace, Pierre-Simon</td>
<td>1</td>
</tr>
<tr>
<td>Lax, Peter David</td>
<td>181</td>
</tr>
<tr>
<td>Lebesgue integral</td>
<td>437</td>
</tr>
<tr>
<td>Lebesgue integral sum</td>
<td>437</td>
</tr>
<tr>
<td>Lebesgue measure</td>
<td>436</td>
</tr>
<tr>
<td>Lebesgue space</td>
<td>440</td>
</tr>
<tr>
<td>in $\mathbb{R}^n$, 16</td>
<td></td>
</tr>
<tr>
<td>in $\mathbb{R}^n$, local, 16</td>
<td></td>
</tr>
<tr>
<td>Length of a path</td>
<td>86</td>
</tr>
<tr>
<td>Levy distribution</td>
<td>134</td>
</tr>
<tr>
<td>Li, Peter</td>
<td>415</td>
</tr>
<tr>
<td>Li-Yau estimate</td>
<td>413</td>
</tr>
<tr>
<td>Liouville theorem</td>
<td>355</td>
</tr>
<tr>
<td>Lipschitz constant</td>
<td>33, 296</td>
</tr>
<tr>
<td>Lipschitz function</td>
<td>33, 296</td>
</tr>
<tr>
<td>Local coordinate system</td>
<td>49</td>
</tr>
<tr>
<td>Locally Lipschitz function</td>
<td>299</td>
</tr>
<tr>
<td>Log-convex function</td>
<td>43</td>
</tr>
<tr>
<td>Lyons, Terry</td>
<td>320, 339</td>
</tr>
<tr>
<td>Markovian properties</td>
<td>123</td>
</tr>
<tr>
<td>Maximum/minimum principle</td>
<td></td>
</tr>
<tr>
<td>elliptic, 230, 293</td>
<td></td>
</tr>
<tr>
<td>elliptic, exterior, 189</td>
<td></td>
</tr>
<tr>
<td>elliptic, in $\mathbb{R}^n$, 13</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Term</th>
<th>Page(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integral maximum principle</td>
<td></td>
</tr>
<tr>
<td>elliptic, exterior, 189</td>
<td></td>
</tr>
<tr>
<td>elliptic, in $\mathbb{R}^n$, 13</td>
<td></td>
</tr>
</tbody>
</table>
for superaveraging functions, 350
parabolic, 223
parabolic, in $\mathbb{R}^n$, 9
strong, elliptic, 229
strong, for superaveraging functions, 350
strong, parabolic, 225, 230
weak, elliptic, 136
weak, parabolic, 138, 141
Maxwell, James Clerk, 2
Maz'ya, Vladimir Gilelevich, 389
McKean, Henry P., Jr., 263
Mean value inequality, 391
Measurable function, 436
Measurable set, 436
on a manifold, 59
Measure
abstract, 434
$\sigma$-finite, 434
Measure space, 437
Mehler kernel, 255, 263, 303
Minakshisundaram S., ix
Minkowski metric, 77
Model manifold with two ends, 240
Mollifier, 18
Monotone convergence theorem, 439
Moser inequality, 371
Moser, Jürgen K., 215, 414
Multiindex, 15, 170
order of, 15
weighted order of, 170
Nash inequality, 368
generalized, 368
Nash, John Forbes, 181, 215, 388
$r$-Neighborhood, 324
Nirenberg, Louis, 181
Null set, 435
Oleinik, Olga Arsen’evna, 320
One point compactification, 141
Parabolic manifold, 313
Parabolic operator, 4, 172
Parseval identity, 432
Partition of unity
in $\mathbb{R}^n$, 19
on a manifold, 52
Perel’man, Grigory Ya., x
Polar coordinates
in $\mathbb{H}$, 77
in $\mathbb{R}^n$, 74
in $\mathbb{S}^n$, 75
on a model manifold, 80
Positive spectrum, 357
Principle of uniform boundedness, 211, 433
Product measure, 442
Product rule, 53, 59, 69
for distributional derivatives, 33
of higher order, 28
for Lipschitz functions, 300
for strong derivatives, 120
for the distributional gradient, 101
in $W^1_0$, 111
Projector, 431
Pullback, 91
Push forward, 92
Push forward measure, 94
Quasi-isometric manifolds, 93
Quasi-isometry, 312
$\mathbb{R}$-differentiation, 53
Radkevich, Evgenii Vladimirovich, 320
Radon-Nikodym derivative, 438
Rayleigh quotient, 272
Regular measure, 436
Relative Faber-Krahn inequality, 409
Rellich theorem, 158
Resolvent, 106, 167, 219
Riemannian manifold, 57
complete, 295
Riemannian measure, 59
Riemannian metric tensor, 56
Riemannian model, 80
Riesz Representation Theorem, 431
Safonov, Mikhail V., 182
Saloff-Coste, Laurent, 389
Schrödinger, Erwin, 2
Schwartz, Laurent-Moïse, 181
Self-adjoint operator, 434
Simple function, 437
Smooth manifold, 50
Sobolev embedding theorem, 151, 214
Sobolev spaces
in $\mathbb{R}^n$, 34, 37
in $\mathbb{R}^{n+1}$, anisotropic, 170
local, 34
on manifolds, 104
Sobolev, Sergei Lvovich, 181
Spectral mapping theorem, 453
Spectral theorem, 452
Stochastic completeness, 231
Stone-Weierstrass theorem, 283
Strichartz, Robert Stephen, 215, 319
Strong derivative, 45
Strong topology, 432
Strongly differentiable, 45
Stroock, Daniel W., 388
Subharmonic function, 229
$\alpha$-Subharmonic function, 229
Submanifold, 70
Subsolution, 391
Sullivan, Dennis Parnell, 363
sup-norm, 16
Superaveraging function, 348
Superharmonic function, 229
$\alpha$-Superharmonic function, 229
Supersolution, 217
Symmetric operator, 168
Täcklind class, 305
Takeda’s inequality, 338
Takeda, Masayoshi, 339
Tangent space, 53
Tangent vector, 53
Test function, 23
Tikhonov class, 305
Tikhonov theorem, 12
Tikhonov, Andrey Nikolayevich, 320
Tonelli’s theorem, 443
Transmutation formula, 121
Ultracontractive semigroup, 365
Ushakov, Vladimir Ignat’evich, 430
Varadhan, Srinivasa R. S., 414
Varopoulos, Nickolas Th., x, 388
Vector field, 56
Volume function, 82
Wave equation, 121, 197
finite propagation speed, 327
Wave operators, 121
Weak compactness, 433
Weak compactness of balls, 433
Weak convergence, 432
Weak derivative, 34
Weak gradient, 99
Weak topology, 432
in $\mathcal{D}'$, 24
Weighted manifold, 67
Weighted model, 82
Weyl’s lemma, 181
Weyl, Hermann Klaus Hugo, 181
Yau, Shing-Tung, x, 319, 320, 363, 415
Titles in This Series

47 Alexander Grigor’yan, Heat Kernel and Analysis on Manifolds, 2009
46.2 Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono, Lagrangian Intersection Floer Theory, 2009
46.1 Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono, Lagrangian Intersection Floer Theory, 2009
45 Lydia Bieri and Nina Zipser, Extensions of the Stability Theorem of the Minkowski Space in General Relativity, 2009
44 Eric Sharpe and Arthur Greenspoon, Editors, Advances in String Theory, 2008
43 Lizhen Ji, Editor, Arithmetic Groups and Their Generalizations, 2008
42.1 Ka-Sing Lau, Zhou-Ping Xin, and Shing-Tung Yau, Editors, Third International Congress of Chinese Mathematicians, 2008
41 Wen-Ching Winnie Li, Editor, Recent Trends in Coding Theory and its Applications, 2007
40 Ovidiu Calin, Der-Chen Chang, and Peter Greiner, Editors, Geometric Analysis on the Heisenberg Group and Its Generalizations, 2007
38 Noriko Yui, Shing-Tung Yau, and James D. Lewis, Editors, Mirror Symmetry V, 2006
37 Lizhen Ji, Jian-Shu Li, H. W. Xu, and Shing-Tung Yau, Editors, Lie Groups and Automorphic Forms, 2006
35 Felix Finster, The Principle of the Fermionic Projector, 2006
34 Ren-Hong Wang, Editor, Computational Geometry, 2003
33 Eric D’Hoker, Duong Phong, and Shing-Tung Yau, Mirror Symmetry IV, 2002
32 Xi-Ping Zhu, Lectures on Mean Curvature Flows, 2002
31 Kiyoshi Igusa, Higher Franz-Reidemeister Torsion, 2002
30 Weiman Han and Mircea Sofonea, Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity, 2002
29 S. T. Yau and Shuxing Chen, Editors, Geometry and Nonlinear Partial Differential Equations, 2002
28 Valentin Afraimovich and Sze-Bi Hsu, Lectures on Chaotic dynamical Systems, 2002
27 M. Ram Murty, Introduction to p-adic Analytic Number Theory, 2002
26 Raymond Chan, Yue-Kuen Kwok, David Yao, and Qiang Zhang, Editors, Applied Probability, 2002
25 Donggao Deng, Daren Huang, Rong-Qing Jia, Wei Lin, and Jian Zhong Wong, Editors, Wavelet Analysis and Applications, 2002
24 Jane Gilman, William W. Menasco, and Xiao-Song Lin, Editors, Knots, Braids, and Mapping Class Groups—Papers Dedicated to Joan S. Birman, 2001
22 Carlos Berenstein, Der-Chen Chang, and Jingzhi Tie, Laguerre Calculus and Its Applications on the Heisenberg Group, 2001
21 Jürgen Jost, Bosonic Strings: A Mathematical Treatment, 2001
19 So-Chin Chen and Mei-Chi Shaw, Partial Differential Equations in Several Complex Variables, 2001
18 Fangyang Zheng, Complex Differential Geometry, 2000
17 Lei Guo and Stephen S.-T. Yau, Editors, Lectures on Systems, Control, and Information, 2000
16 Rudi Weikard and Gilbert Weinstein, Editors, Differential Equations and Mathematical Physics, 2000
15 Ling Hsiao and Zhouping Xin, Editors, Some Current Topics on Nonlinear Conservation Laws, 2000
14 Jun-ichi Igusa, An Introduction to the Theory of Local Zeta Functions, 2000
13 Vasilios Alexiades and George Siopsis, Editors, Trends in Mathematical Physics, 1999
12 Sheng Gong, The Bieberbach Conjecture, 1999
11 Shinichi Mochizuki, Foundations of p-adic Teichmüller Theory, 1999
10 Duong H. Phong, Luc Vinet, and Shing-Tung Yau, Editors, Mirror Symmetry III, 1999
9 Shing-Tung Yau, Editor, Mirror Symmetry I, 1998
8 Jürgen Jost, Wilfrid Kendall, Umberto Mosco, Michael Röckner, and Karl-Theodor Sturm, New Directions in Dirichlet Forms, 1998
7 D. A. Buell and J. T. Teitelbaum, Editors, Computational Perspectives on Number Theory, 1998
6 Harold Levine, Partial Differential Equations, 1997
5 Qi-keng Lu, Stephen S.-T. Yau, and Anatoly Libgober, Editors, Singularities and Complex Geometry, 1997
4 Vyjayanthi Chari and Ivan B. Penkov, Editors, Modular Interfaces: Modular Lie Algebras, Quantum Groups, and Lie Superalgebras, 1997
3 Xia-Xi Ding and Tai-Ping Liu, Editors, Nonlinear Evolutionary Partial Differential Equations, 1997
2.2 William H. Kazez, Editor, Geometric Topology, 1997
2.1 William H. Kazez, Editor, Geometric Topology, 1997
1 B. Greene and S.-T. Yau, Editors, Mirror Symmetry II, 1997
The heat kernel has long been an essential tool in both classical and modern mathematics but has become especially important in geometric analysis as a result of major innovations beginning in the 1970s. The methods based on heat kernels have been used in areas as diverse as analysis, geometry, and probability, as well as in physics. This book is a comprehensive introduction to heat kernel techniques in the setting of Riemannian manifolds, which inevitably involves analysis of the Laplace–Beltrami operator and the associated heat equation.

The first ten chapters cover the foundations of the subject, while later chapters deal with more advanced results involving the heat kernel in a variety of settings. The exposition starts with an elementary introduction to Riemannian geometry, proceeds with a thorough study of the spectral-theoretic, Markovian, and smoothness properties of the Laplace and heat equations on Riemannian manifolds, and concludes with Gaussian estimates of heat kernels.

Grigor’yan has written this book with the student in mind, in particular by including over 400 exercises. The text will serve as a bridge between basic results and current research.