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Issues in Mathematics Education
Volume 7

Research in Collegiate Mathematics Education. III

Alan H. Schoenfeld
Jim Kaput
Ed Dubinsky
Editors

American Mathematical Society
in cooperation with
Mathematical Association of America

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American Mathematical Society
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PREFACE

Welcome to the third volume of *Research in Collegiate Mathematics Education (RCME III)*. For an introduction to the series of volumes see the preface to *RCME I*, which appeared in this *CBMS* series in 1994; for a general introduction to the field of research in undergraduate mathematics education see the first chapter in that volume, by Alan Schoenfeld. In this preface we focus on providing brief introductions to the papers that appear in this volume.

In the three years since we began this enterprise, we have tried to have these volumes grow with the field, reflecting the broad spectrum of work done by researchers concerned with collegiate mathematics education. In journal-like fashion, we consider papers independently and publish those that we think reflect or advance the state of the art. Nevertheless, it still makes sense to structure the collection of papers that have been accepted, for purposes of coherence and to introduce the content and methodological issues they deal with.

In Volumes I and II we established the tradition of bringing each volume to a close with an article focusing on open questions in the field and/or suggestions for future research. Volume I presented Lynn Steen's "Twenty Questions for Research on Undergraduate Mathematics Education," an attempt to "define the challenge and frame the debate" concerning the role of education research in the mathematics enterprise. In Volume II we printed a list of 18 questions raised at the first Oberwolfach conference in Undergraduate Mathematics Education that was held in the Fall of 1995. These are a combination of 18 research and curriculum questions that have no special status other than that after several hours of discussion 24 people at a conference agreed that they are important. In this volume we close with a paper by Annie and John Selden that presents a set of questions that arose during the first annual Conference on Research in Collegiate Mathematics Education held at Central Michigan University, September 5–8, 1996.

The eight papers that comprise the core of this volume can be seen as falling naturally into three sets. The first set contains three papers that focus, in different ways, on the same undergraduate course in mathematical problem solving. The second set also contains three papers, each of which highlights different ways to examine student understanding. The third set contains two reports on how students think about, and try to deal with, the idea of proof in mathematics.

A PROBLEM SOLVING COURSE

This is an unusual series of three papers, each focused on aspects of a problem solving course taught for many years by Alan Schoenfeld and widely reported upon by him and his colleagues. The unusual and highly reflexive nature of the set deserves some explanation. We first received a paper by Abraham Arcavi, Cathy Kessel, Luciano Meira, and Jack Smith, each of whom concentrated on a particular aspect of the classroom activity and Schoenfeld's teaching. While this long four-part paper was in review (a review directed by Kaput and Dubinsky, entirely independently of Schoenfeld) we received a paper by Manuel Santos-Trigo, who likewise analyzed an aspect of the course. All these authors had sat in on the course and/or had analyzed classroom videotapes, transcripts and field notes taken during two offerings of the course (Arcavi et al. in 1990, Santos in 1994). Arcavi et al.'s overall intent was to document and explain the workings of the course's beginnings at a fine level of detail, while at the same time providing sufficient theoretical underpinning to hold the detail together in a principled, explanatory way. Santos deals with related issues, covering more of the course. After receiving positive external reviews that recommended the publication of the two papers contingent upon revisions, and while the two papers were being revised, Schoenfeld was invited to write a response paper in which he reflected on what the others were saying about his teaching. He chose to do so by reviewing the details of their analyses from the perspective of his current work, which is aimed at developing an explicit model of teaching that is intended to explain the real-time decision-making by teachers in terms of their beliefs, knowledge, goals and specific classroom situations. We present the papers in the order of what they cover: Arcavi et al. focusing on the beginning of the course, Santos discussing aspects of the course as a whole, and Schoenfeld in response to both.

This set of papers is intended to provide a close look at a particular example of "good practice," a highly refined course and pedagogical approach that over the years seems to succeed in teaching powerful problem solving skills. It is not intended to be construed as the only form of good practice, or even as a model for everyone to emulate; many other instructors have their own successful ways of teaching their version of problem solving. We hope, however, that these papers are seen as being of value for substantive and methodological reasons. They examine a good practice in fine-grained detail, exploring how real-time classroom decision-making is related to the instructor's knowledge, goals and beliefs about problem solving, and suggesting how it may be taught and learned. Moreover, they serve as case studies in how to document and analyze classroom situations.

Classroom practice, both good and bad, merely seems to happen—somewhat like the weather. While it is often discussed in general and even passionate terms, and people struggle to measure its outcomes, it is seldom examined in a detailed way, where claims are explicitly linked to events, and a coherent explanatory story is put together. Our ultimate goal for studies of this type is that they lead to a new level of analytic explicitness regarding the varieties of classroom practice that will help expose the underpinnings of good teaching so that more of us can achieve it more easily and routinely.

METHODOLOGICAL PLURALISM

Marilyn Carlson applies a combination of quantitative and qualitative methods to the study of understanding the function concept. Carlson has designed a written examination that helps her in the selection of students for more extensive (and time-consuming) follow-up interviews. Written examinations can be administered to large numbers of students, providing "large n" data and enough information about individual students to sort them into rough equivalence classes regarding some aspects of their mathematical understandings. One can then interview representatives from those classes, reducing the interviews to a manageable number while still exploring a wide range of student understandings. Also, the interviews provide an opportunity for triangulation on the findings from the written exam—that is, for seeing whether what the students do in interviews corresponds to the inferences one can draw from their written work. Aside from the question of methodology are Carlson's findings. She considered high performing students across a range of undergraduate levels from those taking college algebra to students beginning graduate work in mathematics. Unfortunately, she finds that difficulties in developing a powerful function concept occur at all levels and are even seen in our brightest students. One thing this paper does is point to the complexity of the concept, and the multiple sets of understandings one must develop in coming to grips with it. It also shows that, in spite of the tremendous amount of research on understanding the function concept and the important improvements in pedagogy that it has led to, much still remains to be done. As Carlson concludes from the study, "an individual's view of the function concept evolves over a period of many years and requires an effort of 'sense making' to understand and orchestrate individual function components to work in concert."

David Meel used a similarly pluralistic methodology to analyze students' understandings of selected calculus concepts (limit, derivative, and integral) by students in a third-semester honors calculus class. But there are important differences in the way Meel analyzed his data. While Carlson's selection criteria for student interviews were grounded in the "at-a-point vs. across-time" perspective of Steve Monk as well as the APOS Theory of the RUMEC group, Meel employed a specific model of understanding developed by Pirie and Kieren, together with standard statistical methods, to largely direct his analysis of the data. While addressing broader issues of conceptual understanding in general, Meel is also concerned with a more focused issue: the study of the relative effects of two very different pedagogical approaches to calculus, a reformed course called *Calculus &Mathematica* and the traditional calculus course. Meel compares students from these courses in terms of how well they can perform calculations, their problem-solving ability, and their level of conceptual understanding. The results of the comparison are mixed. Meel finds no serious differences between the two groups in their overall performance on the written instrument and in particular with respect to the differentiation concept, procedurally-oriented items and text-and-pictorial items. He does, however find that the students taking a traditional course performed significantly better on the limit concept, conceptually-oriented items, and on text-only items. On the problem-solving interviews, however, he finds that the *Calculus&Mathematica* students tended to be more flexible and were more successful in solving problems. In additional interviews which he calls "understanding interviews" he finds little difference between the two groups. Moreover, he finds serious inadequacies in both groups with respect to understanding the limit and differentiation concepts, although they did much better with integration. Meel's overall conclusion seems to be a concern with whether "either curriculum satisfied the needs of these honors calculus students."

The methodological challenges faced by Alvin Baranchik and Barry Cherkas were very different indeed. The question they faced was this. Suppose you have available a widely used standardized test for elementary algebra—in their case the College Entrance Examination Board's *Elementary Algebra Skills Test*. The test uses multiple choice items, and reports the number of items the student has gotten correct. What kinds of inferences can you draw about students' understandings, if you look at the full pattern of their responses to the test? Baranchik and Cherkas had experts evaluate responses to the test items (including "distracter" items) for clues to possible understandings, and they assigned partial credit for such partial understandings; they also used statistical techniques to identify components of algebraic skills from patterns of responses to the items. The two methods agreed, and provided some predictive power. The bottom line of the study: "Genuine partial understanding, which is relevant to student performance and is missed by number correct scoring alone, can be inferred from students' selections of certain incorrect alternatives."

PROOFS IN MATHEMATICS

In this invited foundational paper Guershon Harel and Larry Sowder provide a distinctly psychological framework (although informed by historical, philosophical and cultural analyses) for examining students' understanding of proof. The paper was developed as part of the first author's NSF funded research on students' understanding, production and appreciation of mathematical proofs, ultimately focusing on mathematics majors as the target population. Building on a literature review, Harel and Sowder look closely at what constitutes evidence in the eye of the student rather than in the eye of the instructor or curriculum writer, and they expose deep differences that help to explain the historical failure of most students to learn the hows and whys of mathematical proof. Their main strategy is as follows: to characterize students' cognitive schemes of proof—how students understand proof, evidence and justification; to document how these schemes are reflected in the behavior of college students working in different domains and at different levels; to show how these schemes develop over time and experience among mathematics majors; and to offer principles for instruction that help facilitate appropriate development. They examine evidence taken from a series of teaching experiments executed by the first author in Number Theory, Geometry, introductory and advanced Linear Algebra, and a case study of a precocious junior high school student taking Geometry and Calculus at the university level. The paper is not about how to teach proof, or about comparative methods (Method A versus Method B). Instead, by closely examining student behaviors and responses to a variety of teaching experiments, it tries to construct a systematic cognitivist way of understanding how students understand and learn how to prove mathematical assertions. The resulting system of *External-Conviction*, *Empirical* and *Analytic Proof Schemes*, each with several interconnected sub-schemes, is not simple (we would have every reason to be suspicious if it were), but it is consistent with a substantial body of episodic evidence garnered from the teaching experiments. The lengthy series of episodes taken from the teaching experiments illustrate each type of proof scheme that they describe and relate it to the work of others in the field. However, as the authors note, they do not regard their typology as the last word on the subject. Rather, it should be seen as a starting point that will need refinement and data from many other sources as well as more sequenced teaching experiments to examine the longer term development of these schemes. Nonetheless, they suggest how we might begin to examine how existing practices foster certain weaknesses, how instructors can recognize and interpret their students' proof behaviors, and how alternative practices and curricula might build more powerful schemes.

David Gibson takes a more focused look at one aspect of how students develop proofs and what teaching can do to help. Rather than consider the whole question of proofs as did Harel and Sowder, Gibson looks at the method of using diagrams in the course of proving a statement in mathematics. He conducted interviews of 12 students who had just completed a first course in analysis, discussing with them the proofs they gave on three separate homework problems. Gibson describes what students use diagrams for, how these diagrams might help students, and why they seemed to help. He found four things students used diagrams for and analyzes the interviews to find out something about the how and the why. Gibson finds that diagrams were used to help students understand given information. The diagrams seemed to appeal to students' natural thinking, partly because the diagrams were concrete and reduced mental load. Students used diagrams to judge the truthfulness of statements. Diagrams helped students apply their criteria for truth; and this was because the diagrams corresponded to the students' criteria. Making discoveries was stimulated by diagrams. They helped students obtain ideas more readily, and the students reported that diagrams corresponded to their understanding of information, were alterable and reduced mental burden. Finally, diagrams were used to write out ideas. They helped link the student's ideas to verbal and symbolic representation. Again, this was because diagrams were seen to correspond to ideas and reduce mental load. Gibson argues that information of this sort could be helpful to teachers in planning their pedagogy for helping students learn to make proofs.

QUESTIONS

At a 1996 conference that was organized by the Research in Undergraduate Mathematics Education Community (RUMEC) and sponsored by the Exxon Educational Foundation, a number of research questions emerged out of the presentation and discussion of research reports. John and Annie Selden selected some of those questions and organized them for presentation here. Open questions are the heart of a research enterprise. Among the various ways in which we can decide what is an important problem to work on or evaluate a piece of research is to look for problems that others have raised, have worked on, and would like to see progress on. It is a way of making more objective judgments and raising the quality levels of the work that we do. The Seldens worked to identify the questions that people were interested in. They selected a total of 30 questions, which they organized in four categories: questions about the nature of mathematics and its teaching and learning; questions (which could be refined for) research; questions about RUME and how one does it; and questions asked by practitioners (college mathematics teachers) to members of the research community. It is certainly the case that we can watch our field grow and mature by observing an increasing concern with trying to solve problems that others are posing.

Alan Schoenfeld
Jim Kaput
Ed Dubinsky

RESEARCH IN COLLEGIATE MATHEMATICS EDUCATION

EDITORIAL POLICY

The papers published in these volumes will serve both pure and applied purposes, contributing to the field of research in undergraduate mathematics education and informing the direct improvement of undergraduate mathematics instruction. The dual purposes imply dual but overlapping audiences and articles will vary in their relationship to these purposes. The best papers, however, will interest both audiences and serve both purposes.

CONTENT.

We invite papers reporting on research that addresses any and all aspects of undergraduate mathematics education. Research may focus on learning within particular mathematical domains. It may be concerned with more general cognitive processes such as problem solving, skill acquisition, conceptual development, mathematical creativity, cognitive styles, etc. Research reports may deal with issues associated with variations in teaching methods, classroom or laboratory contexts, or discourse patterns. More broadly, research may be concerned with institutional arrangements intended to support learning and teaching, e.g. curriculum design, assessment practices, or strategies for faculty development.

METHOD.

We expect and encourage a broad spectrum of research methods ranging from traditional statistically-oriented studies of populations, or even surveys, to close studies of individuals, both short and long term. Empirical studies may well be supplemented by historical, ethnographic, or theoretical analyses focusing directly on the educational matter at hand. Theoretical analyses may illuminate or otherwise organize empirically based work by the author or that of others, or perhaps give specific direction to future work. In all cases, we expect that published work will acknowledge and build upon that of others—not necessarily to agree with or accept others' work, but to take that work into account as part of the process of building the integrated body of reliable knowledge, perspective and method that constitutes the field of research in undergraduate mathematics education.

REVIEW PROCEDURES.

All papers, including invited submissions, will be evaluated by a minimum of three referees, one of whom will be a Volume editor. Papers will be judged on the basis of their originality, intellectual quality, readability by a diverse audience, and the extent to which they serve the pure and applied purposes identified earlier.

SUBMISSIONS.

Papers of any reasonable length will be considered, but the likelihood of acceptance will be smaller for very large manuscripts.

Five copies of each manuscript should be submitted. Manuscripts should be typed double-spaced, with bibliographies done in the style established by the American Mathematical Society for its CBMS series of volumes (an example style sheet is available from the editors on request).

Note that the *RCME* volumes are produced for electronic submission to the AMS. Accepted manuscripts should be prepared using AMS-TeX 2.1 (the macro packages are available through e-mail without charge from the AMS). Illustrations should also be prepared in a form suitable for electronic submission (namely, encapsulated postscript files).

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