Number 2

Gilbert Baumslag

Lecture notes on nilpotent groups

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by

Gilbert Baumslag

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Introduction

These lectures are concerned, in the main, with finitely generated nilpotent groups. The theory of these groups is rich and exciting.

There seem to be three main parts to this theory. The first of these deals with the so-called commutator calculus, which was initiated by Philip Hall in his fundamental paper [29]. As the name suggests it is concerned with manipulations of commutators and deductions of further relationships between commutators from basic identities. There does not seem to be any guiding principle in this calculus; consequently this aspect of the theory is for the most part rather difficult.

The second aspect of the theory is, in a sense, governed by a single principle. This principle may be likened to a well-known procedure in elementary number theory where one shows that a proposition about the integers holds modulo each prime \( p \) and thence for the integers themselves. This notion may be used, in particular, to prove certain results about finitely generated torsion-free nilpotent groups. The key fact is the following theorem of K. W. Gruenberg [25]: if \( G \) is a finitely generated torsion-free nilpotent group and \( p \) is any prime, then, given any element \( g \in G \ (g \neq 1) \), there is a normal subgroup \( N \) of \( G \) such that \( G/N \) is a finite \( p \)-group with \( g \not\in N \). Roughly speaking, the idea is to show that if a proposition about a finitely generated torsion-free nilpotent group holds for all its homomorphic images of prime-power order, then it holds also for the group itself.

The third part of nilpotent group theory stems in the main from the connection between lie groups and lie algebras; it was discussed first by A. I. Mal'cev in his beautiful paper [63]. The impact of this connection and the consequent connections between arithmetic and algebraic groups has only very recently emerged (see, for example, the very deep papers by L. Auslander [2], [3] and the paper by L. Auslander and G. Baumslag [4]). In a sense this is the most exciting, although it is in some ways the most limited, aspect of the whole theory. Here we shall develop \textit{ab initio}, by using the approach of S. A. Jennings [47], as much of the necessary machinery as is needed for our discussion of the automorphism groups of finitely generated nilpotent groups.

Although I have chosen to divide the theory of finitely generated nilpotent groups into three parts, it should be pointed out that these parts are really very much inter-related, each complementing the others.

The program for these lectures is set forth in the table of contents. I have not
divided the material up into the three parts described above because this would make for very awkward exposition. I have also not tried to provide a complete account of the theory here. The interested reader will find that a study of the bibliography included at the end of these lectures will allow him to go more deeply into those aspects of the theory which appeal to him most.

Acknowledgement

These notes are substantially the same as those prepared as an aid to the ten lectures on finitely generated nilpotent groups which I gave in Austin at the University of Texas during May of 1969. They have been slightly polished and numerous mistakes have been eradicated, mainly due to the diligence of John F. Ledlie to whom I would like to express my appreciation for his valuable help and assistance. I would also like to express my thanks to Nancy Singleton and Kathy Vigil for their patient deciphering of many rewritten, half legible pages which made up this manuscript. Finally I would like to thank the Mathematics Department (especially John R. Durbin) of the University of Texas for making it such a pleasure to spend a week in May in Austin.
Theorem 5.10. Every solvable group of matrices over the ring of integers of an algebraic number field is polycyclic.

Theorem 5.10 yields two further theorems of A. I. Mal'cev [62] namely:

Theorem 5.11. The solvable subgroups of the group of automorphisms of a polycyclic group are polycyclic.

Theorem 5.12. A solvable group is polycyclic if and only if all its abelian subgroups are finitely generated.

We turn our attention now to a rather different notion due to J. Milnor [67] which essentially singles out the nilpotent groups from other solvable groups. Milnor's notion is that of growth function. In order to explain let $G$ be a group with a given finite set $X = \{x_1, \ldots, x_q\}$ of generators. For each positive integer $n$ let $g(n)$ denote the number of elements of $G$ that can be expressed as words of length at most $n$. As it stands $g(n)$ is a function of the positive integer $n$. The function $g$ is called a growth function for $G$. We term $g$ of polynomial type of degree $\leq e$ if $g(n) \leq cn^e$ where $c$ is some constant. If $g(n) \geq uv^n$ where both $u$ and $v$ are constants, $v > 1$, then $g$ is said to be of exponential type. It turns out that if any growth function of $G$ is of a given type, then they all are, i.e., the choice of the finite system of generators of $G$ does not affect the type of the associated growth function. The relevance of these notions to us lies in the following result due in part to J. Milnor and in part to J. A. Wolf (see [87]).

Theorem 5.13. A finitely generated solvable group is either of exponential type or of polynomial type. It is of polynomial type if and only if it is a finite extension of a nilpotent group.

The relevance of this notion of a growth function to algebra is not clear, but it is still quite fascinating. It arose in connection with the study of curvature of certain Riemannian manifolds (see J. Milnor [68] and J. A. Wolf [87]).

BIBLIOGRAPHY


