Conference Board of the Mathematical Sciences
REGIONAL CONFERENCE SERIES IN MATHEMATICS

Number 3

Lawrence Markus

Lectures in differentiable dynamics

Revised Edition

Supported by the National Science Foundation
Published by the American Mathematical Society

LECTURES IN DIFFERENTIABLE DYNAMICS
Revised Edition

This page intentionally left blank

http://dx.doi.org/10.1090/cbms/003

Conference Board of the Mathematical Sciences

REGIONAL CONFERENCE SERIES IN MATHEMATICS

supported by the

National Science Foundation

Number 3

LECTURES IN DIFFERENTIABLE DYNAMICS
Revised Edition

by

Lawrence Markus

Published for the

Conference Board of the Mathematical Sciences

by the

American Mathematical Society

Providence, Rhode Island

Expository Lectures
from the CBMS Regional Conference
held at Case Western Reserve University, Cleveland, Ohio
June 2–6, 1969

International Standard Serial Number 0160-7642
International Standard Book Number 0-8218-1695-0
AMS 1970 Subject Classifications: 3402, 34C35,
54H20, 58F10, 58F15, 7002, 70G35.
Key Words and Phrases: dynamical systems,
topological dynamics, nonlinear mechanics, diffeomorphism,
differentiable manifold, structural stability, Morse-Smale,
Anosov hyperbolic, generic properties.

Library of Congress Cataloging in Publication Data

Markus, Lawrence.

Lectures in differentiable dynamics.

(Regional conference series in mathematics; no. 3)

"Expository lectures from the CBMS Regional Conference held at Case Western Reserve University, Cleveland, Ohio, June 2–6, 1969."

Bibliography: p.

1. Differential equations. 2. Topological dynamics. 3. Differentiable dynamical systems. I. Conference Board of the Mathematical Sciences. II. Title. III. Series.

QA1.R33 no. 3, 1980 [QA372] 510s [515.3'5] 80-16847

CONTENTS

Abstract

These Lectures provide a survey of the modern theory of differentiable dynamics as an abstraction of the qualitative theory of ordinary differential equations. Historical and conceptual developments are emphasized as the theories of nonlinear mechanics, topological dynamics, and differential topology contribute to the formation of differentiable dynamics. Important classes of dynamical systems, such as structurally stable, Morse-Smale, Anosov hyperbolic, and generic systems, are described and related to one another and to nonlinear mechanics.

$$K(v, \hat{v}, \Psi) = \Psi v - \hat{v}\Psi$$

on the Banach space $C^1(T^n) \times C^1(T^n) \times C^0(T^n)$ into $C^0(T^n)$. We seek to solve the equation $K = 0$ for Ψ near the initial data

$$v: x \longrightarrow Ax, \ \hat{v}: x \longrightarrow Ax, \ \Psi = \text{identity}.$$

It is not difficult to prove that the function K is in class C^1, and has a bounded invertible operator for the partial derivative with respect to Ψ, at the initial data (Ax, Ax, id). Then the existence of the continuous map $\Psi \in C^0(T^n)$, for each v and $\hat{v}$ near $x \longrightarrow Ax$, follows from the implicit function theorem. A further argument proves that Ψ is the required homeomorphism of T^n onto itself. This method of proof of Anosov's theorem was suggested by Mather [39].

These diverse theorems on Hyp, MS, SS, and gen(1, 2, 3, 4) systems constitute the principal theorem of differentiable dynamics, as presented at the beginning of this section.

BIBLIOGRAPHY

1. A. Andronov and L. Pontryagin, *Systèmes grossiers*, C. R. (Dokl.) Acad. URSS 14 (1937), 247–251.

2. D. V. Anosov, *Roughness of geodesic flows on compact Riemannian manifolds of negative curvature*, Dokl. Akad. Nauk SSSR 145 (1962), 707–709 = Soviet Math. Dokl. 3 (1962), 1068–1069. MR 26 #716.

3. ———, *Geodesic flows on closed Riemannian manifolds of negative curvature*, Trudy Mat. Inst. Steklov. 90 (1967),= Proc. Steklov. Inst. Math. 90 (1967), MR 36 #7157.

4. V. I. Arnol'd and Ja. G. Sinaĭ, *On small perturbations of the automorphisms of a torus*, Dokl. Akad. Nauk SSSR 144 (1962), 695–698 = Soviet Math. Dokl. 3 (1962), 783–786. MR 26 #6535.

5. L. Auslander, F. Hahn and L. Markus, *Topological dynamics on nilmanifolds*, Bull. Amer. Math. Soc. 67 (1961), 298–299. MR 23 #A636.

6. L. Auslander, L. Green, F. Hahn, L. Markus, et al. *Flows on homogeneous spaces*, Ann. of Math. Studies, no. 53, Princeton Univ. Press, Princeton, N. J., 1963. MR 29 #4841.

7. L. Auslander and R. E. Mackenzie, *Introduction to differentiable manifolds*, Mc-Graw-Hill, New York, 1963. MR 28 #4462.

8. G. D. Birkhoff, *Dynamical systems*, Amer. Math. Soc. Colloq. Publ., vol. 9, Amer. Math. Soc., Providence, R. I., 1927.

9. N. N. Bogolyubov and Yu. A. Mitropolskiĭ, *Asymptotic methods in the theory of non-linear oscillations*, GITTL, Moscow, 1955; 2nd ed., Fizmatgiz, Moscow, 1958; English transl. of 2nd ed., Gordon and Breach, New York, 1962. MR 17, 368; MR 20 #6812.

10. E. Cartan, *La géométrie des espaces de Riemann,* Gauthier-Villars, Paris, 1925.

11. M. L. Cartwright and J. E. Littlewood, *On non-linear differential equations of the second order.* II, Ann. of Math. (2) 48 (1947), 472–494; erratum, ibid. (2) 49 (1948), 1010. MR 9, 35; MR 10, 121.

12. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations,* McGraw-Hill, New York, 1955. MR 16, 1022.

13. H. F. DeBaggis, *Dynamical systems with stable structure,* Contributions to the Theory of Nonlinear Oscillations, vol. 2, Princeton Univ. Press, Princeton, N. J., 1952, pp. 37–59. MR 14, 557.

14. A. Denjoy, *Sur les courbes définies par les équations différentielles à la surface du tore,* J. Analyse Math. 11 (1932), 333–375.

15. S. P. Diliberto, *Perturbation theorems for periodic surfaces.* I. *Definitions and main theorems.* I, II, Rend. Circ. Mat. Palermo (2) 9 (1960), 265–299; ibid. (2) 10 (1961), 111–161. MR 26 #420a, b.

16. N. Fenichel, *Perturbation theory of invariant manifolds of vector fields,* Thesis, New York University, 1970.

17. G. Floquet, *Sur les équations différentielles linéaires à coefficients periodiques,* Ann. Ecole Norm. Sup. (2) 12 (1883), 47–89.

18. W. H. Gottschalk and G. A. Hedlund, *Topological dynamics,* Amer. Math. Soc. Colloq. Publ., vol. 36, Amer. Math. Soc., Providence, R. I., 1955. MR 17, 650.

19. L. Green, *Geodesic flows,* Proc. Sympos. Warwick, 1969, Springer. (To appear).

20. J. Hale, *Ordinary differential equations,* Wiley, New York, 1969.

21. P. Hartman, *Ordinary differential equations,* Wiley, New York, 1964. MR 30 #1270.

22. C. Hayashi, *Nonlinear oscillations in physical systems,* McGraw-Hill Electrical and Electronic Engrg. Ser., McGraw-Hill, New York, 1964. MR 30 #312.

23. W. Kaplan, *Regular curve-families filling the plane,* I, II, Duke Math. J. 7 (1940), 154–185; ibid. 8 (1941), 11–46. MR 2, 322.

24. A Katok, *On the existence of smooth ergodic flows* (see Appendix).

25. L. Kronecker, *Werke.* Band III, Teubner, Leipzig, 1899.

26. I. Kupka, *Contributions à la théorie des champs génériques,* Contribution to Differential Equations 2 (1963), 457–484. MR 29 #2818a.

27. J. Lagrange, *Oeuvres.* I, Paris, 1762, p. 520.

28. S. Lefschetz, *Differential equations: Geometric theory,* Interscience, New York, 1957. MR 20 #1005.

29. N. Levinson, *Small periodic perturbations of an autonomous system with a stable orbit,* Ann. of Math. (2) 52 (1950), 727–738. MR 12, 335.

30. A. Liapunov, *Problème général de la stabilité du mouvement,* Kharkov, 1892; French transl., Ann. Fac. Sci. Univ. Toulouse (2) 9 (1907), 203–474; reprint, Ann. of Math. Studies, no. 17, Princeton Univ. Press, Princeton, N. J., 1947. MR 9, 34.

31. L. Markus, *Global structure of ordinary differential equations in the plane,* Trans. Amer. Math. Soc. 76 (1954), 127–148. MR 15, 704.

32. ———, *Invariant sets of structurally stable differential systems,* Proc. Nat. Acad. Sci. U.S.A. 45 (1959), 1633–1634.

33. ———, *Periodic solutions and invariant sets of structurally stable differential systems,* Bol. Soc. Mat. Mexicana (2) 5 (1960), 190–194. MR 26 #1561.

34. ———, *Sur l'équations différentielles avec la stabilité structurelle,* Proc. Internat. Conference of Dynamical Systems, Cime Roma, 1960, pp. 1–7.

35. ———, *Structurally stable differential systems,* Ann. of Math. (2) 73 (1961), 1–19. MR 24 #A2724.

36. ———, *Generic properties of differential equations,* Proc. Internat. Sympos. Nonlinear Differential Equations and Nonlinear Mechanics, Academic Press, New York, 1963, p. 22.

37. ———, *The misbehavior of the solutions of a differential system near a periodic solution,* Proc. Conference Qualitative Theory, Madison, Wis., 1968.

38. L. Markus and Meyer, *Generic Hamiltonian systems are not ergodic,* Proc. Sympos., Kiev, 1969.

39. J. Mather, *Appendix to part* I: *Anosov diffeomorphisms,* of S. Smale, "Differentiable dynamical systems", Bull. Amer. Math. Soc. 73 (1967), 792–795.

40. N. Minorsky, *Introduction to non-linear mechanics. Topological methods. Analytical methods. Non-linear resonance. Relaxation oscillations,* Edwards, Ann Arbor, Mich., 1947. MR 8, 583.

41. Yu. A. Mitropolskiĭ, *On the investigation of an integral manifold for a system of nonlinear equations with variable coefficients,* Ukrain. Mat. Ž. 10 (1958), 270–279. MR 21 #176a.

42. J. Moser, *On a theorem of Anosov,* J. Differential Equations 5 (1969), 411–440. MR 38 #6633.

43. V. V. Nemyckiĭ and V. V. Stepanov, *Qualitative theory of differential equations,* 2nd ed., GITTL, Moscow, 1949; English transl., Princeton Math. Series, no. 22, Princeton Univ. Press, Princeton, N. J., 1960. MR 22 #12258.

44. J. C. Oxtoby and S. M. Ulam, *Measure-preserving homeomorphisms and metrical transitivity,* Ann. of Math. (2) 42 (1941), 874–920. MR 3, 211.

45. J. Palis and S. Smale, *Structural stability theorems,* Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R. I., 1970, pp. 223–231.

46. M. M. Peixoto, *On structural stability,* Ann. of Math. (2) 69 (1959), 199–222. MR 21 #753.

47. ———, *Structural stability on two-dimensional manifolds,* Topology 1 (1962), 101–120. MR 26 #426.

48. H. Poincaré, *Les méthodes nouvelles de la mécanique céleste.* Vols. I, II, III, Paris, 1892, 1893, 1899; reprint, Dover, New York, 1957. MR 19, 414.

49. C. C. Pugh, *An improved closing lemma and a general density theorem,* Amer. J. Math. 89 (1967), 1010–1021. MR 37 #2257.

50. ———, *Linearization near an invariant set*, Proc. Sympos. Warwick, 1969. Springer (see Appendix).

51. R. J. Sacker, *A perturbation theorem for invariant Riemannian manifolds*, Proc. Internat. Sympos Differential Equations and Dynamical Systems (Mayaguez, P. R., 1965) Academic Press, New York, 1967, pp. 43–54. MR 36 #1784.

52. A. J. Schwartz, *A generalization of a Poincaré-Bendixson theorem to closed two-dimensional manifolds*, Amer. J. Math. 85 (1963), 453–458, 753. MR 27 #5003.

53. H. Seifert, *Closed integral curves in 3-space and isotopic two-dimensional deformations*, Proc. Amer. Math. Soc. 1 (1950), 287–302. MR 12, 273.

54. Ja. G. Sinaĭ, *Probabilistic concepts in ergodic theory*, Proc. Internat. Congress Math. (Stockholm, 1962) Inst. Mittag-Leffler, Djursholm, 1963, pp. 540–559; English transl., Amer. Math. Soc. Transl. (2) 31 (1963), 62–84. MR 32 #196.

55. S. Smale, *Morse inequalities for a dynamical system*, Bull. Amer. Math. Soc. 66 (1960), 43–49. MR 22 #8519.

56. ———, *On gradient dynamical systems*, Ann. of Math. (2) 74 (1961), 199–206. MR 24 #A2973.

57. ———, *A structurally stable differentiable homeomorphism with an infinite number of periodic points*, Proc. Internat. Sympos. Non-Linear Vibrations (1961), vol. II, Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1963, pp. 365–366. MR 28 #3434.

58. ———, *Dynamical systems and the topological conjugacy problem for diffeomorphisms*, Proc. Internat. Congress Math. (Stockholm, 1962), Inst. Mittag-Leffler Djursholm, 1963, pp. 490–496. MR 31 #759.

59. ———, *Diffeomorphisms with many periodic points*, Differential and Combinatorial Topology, Princeton Univ. Press, Princeton, N. J., 1965, pp. 63–80. MR 31 #6244.

60. ———, *Structurally stable systems are not dense*, Amer. J. Math. 88 (1966), 491–496. MR 33 #4911.

61. ———, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. 73 (1967), 747–817. MR 37 #3598.

62. N. Steenrod, *Topology of fibre bundles*, Princeton Math. Series, vol. 14, Princeton Univ. Press, Princeton, N. J., 1951. MR 12, 522.

63. S. Sternberg, *Lectures on differential geometry*, Prentice-Hall, Englewood Cliffs, N. J., 1964. MR 33 #1797.

64. F. W. Wilson, Jr., *On the minimal sets of non-singular vector fields*, Ann.of Math. (2) 84 (1966), 529–536. MR 34 #2028.

This page intentionally left blank

APPENDIX (1980)

Scope and Structure of the Appendix

During the past decade the discipline of differentiable dynamics has flourished, branched, and borne fruit with hot-house intensity. The range of scientific activity is demonstrated by the many major symposia and conferences in various subfields and related areas:

I. *Nonlinear Oscillation,* with motivations and applications from engineering [Conference References: 2, 4, 5, 7, 16, 21, 23, 24, 31, 32, 34, 35, 36, 38, 40, 41, 43] and from diverse sciences such as astronomy, biology, ecology, and economics [Conference References: 9, 11, 12, 13, 14, 26, 27, 30, 31, 38, 40, 41, 42, 43].

II. *Diffeomorphisms and Foliations,* including special constructions and examples of significant interest on particular low dimensional spaces [Conference References: 10, 19, 20, 22, 25, 28, 33, 44].

III. *General Theory–Dissipative Dynamics* (including Axiom A systems) [Conference References: 1, 2, 3, 5, 7, 8, 10, 14, 16, 17, 20, 22, 23, 24, 25, 28, 29, 33, 34, 35, 36, 39, 41, 44].

IV. *General Theory–Conservative Dynamics* (including Hamiltonian systems) [Conference References: 1, 2, 3, 5, 7, 8, 17, 19, 20, 25, 28, 32, 33, 34, 35, 36, 37, 40, 41, 44].

V. *Chaos, Catastrophe, and Multi-Valued Trajectories* (including parts of ergodic theory; and some topics in the related dynamics of control and polysystems, stochastic differential systems, and functional-differential systems) [Conference References: 4, 6, 11, 12, 13, 14, 15, 17, 18, 20, 23, 26, 27, 29, 30, 31, 32, 33, 37, 38, 39, 40, 42, 43, 44].

We introduce the prior five headings I–V, with the corresponding conference listings, as an organizational base for this appendix. In fact, the given outline may turn out to be the most valuable part of this appendix, since, at best, the appendix consists of sketchy descriptions of some selected important results, and some fragmentary mentions of some selected major theories. We shall not attempt to specify every last technical hypothesis in each theory, but shall treat this appendix more as an annotated bibliography indicating the direction and nature of the research in differentiable dynamics since the first publication of this monograph ten years ago. The new References include Conferences, Surveys, and Articles.

Besides the astonishing frenzy of activity evidenced by the multitude of symposia, even more astonishing is the recognition that a few of the challenging problems posed in the original monograph have actually been resolved [25, 41, 80, 92, 107]. But, as in all mathematical development, the phrasing of new problems and questions, and the organization of new theoretical methods to treat them, form the central effort of the research in differentiable dynamics over the past decade. In particular we should emphasize the incorporation of the powerful KAM results into the general framework of conservative dynamics as in area IV, and the new research area V.

The original edition of this monograph still serves as a useable conceptual, organizational, and notational basis for differentiable dynamics. This appendix will serve only to update the results in areas I, II, III; and as a guide to provide a brief introduction to areas IV and V not covered in the monograph.

I. Nonlinear Oscillations

Instead of the hopeless and thankless task of attempting to outline and organize the torrent of contributions to the theory of nonlinear oscillations (mainly for second order differential equations), we shall merely comment on several results bearing on the examples and methods mentioned in the original monograph. For a key to the extensive literature in this broad field we refer to the proceedings of the appropriate symposia and conferences, as indicated previously under area I.

The damped nonlinear oscillator, with one degree of freedom (see this monograph, page 7),

$$\ddot{x} + f(x)\dot{x} + g(x) = e(t)$$

has frictional coefficient $f(x) > 0$, restoring force $g(x)$ (so $xg(x) > 0$ for $x \neq 0$), and periodic driving force $e(t + P) \equiv e(t)$. A new result [112] asserts that

$$\ddot{x} + b\dot{x} + g(x) = e(t)$$

has a unique periodic response, for each sufficiently large constant $b > 0$. The same result might be expected for a frictional coefficient $bf(x) > 0$ (compare page 10).

For the frictionless forced oscillator

$$\ddot{x} + g(x) = e(t),$$

it is of interest to show that each solution remains bounded in the $(x, y = \dot{x})$ phase plane (see page 11). This has been demonstrated [80] for $g(x) = x^3$, $e(t) = \sin t$ but the method, which shows that the Poincaré map of the phase plane is an area-preserving "twist map" around infinity (with corresponding invariant curves encircling infinity), probably applies to more general cases.

There have been serious attempts [59] to study the Poincaré map for the Duffing oscillator, with $g(x) = x + \epsilon(2\alpha x + 4\beta x^3)$, $e(t) = -\epsilon\alpha \cos t$, $\epsilon > 0$ small, to find homoclinic orbits. It is well known [121] that a transverse intersection of attracting-repelling curves (a transverse homoclinic orbit) implies the existence of infinitely many long period orbits. The main difficulty has been to prove the desired transversality—but this has been accomplished [19] for certain frictionless forced oscillators, for some orbits in celestial mechanics [2, 84], and recently by M. Levi in an analysis of the forced van der Pol equation [51] (brilliantly clarifying the early classical work of Cartwright and Littlewood).

An old technique, but now much more powerful because of the improvement in computer facilities, is the numerical and graphical exploration of the phase-plane portrait and the Poincaré map of the phase-plane. These computer explorations, reinforced by mathematical estimates, can be used to find topological indices for fixed points—and hence for periodic orbits of the corresponding dynamical system. Such methods have been applied very successfully [106, 138] to the frictionless forced Duffing oscillator.

Some years ago the concepts of nonstandard analysis were introduced explicitly into topological dynamics to define limit sets on noncompact phase spaces. More recently the methods of nonstandard analysis have been used to study particular dynamical systems, for instance, the van der Pol equation,

$$\epsilon\ddot{x} + (x^2 - 1)\dot{x} + x = A.$$

When $\epsilon \longrightarrow 0$ the unique periodic solution has the form of a "relaxation oscillation". Within nonstandard analysis we can let ϵ be an infinitesimal number, study the corresponding nonstandard periodic solution, and then relate this approach to the classical analysis of the relaxation oscillation [17, 21].

In the Newtonian celestial mechanics of three bodies new types of periodic oscillations have been studied [2, 56]. Other very important contributions to the dynamics of the n-body problem clarify and analyse the nature of the possible collisions and singularities [57, 58, 105].

Within mathematical biology the classical areas such as diffusion in biochemistry, and fluid dynamics within the circulatory systems of physiology have been pursued strongly. But also new directions have arisen with new models of population dynamics and epidemiology being proposed—although these models often involve retarded information about earlier generations, and so are formulated as differential-delay systems (see V). Another new direction concerns mathematical ecology, where an interesting example displays an ecological model [7] with several species cyclically occupying the same niche. This example surprised many ecologists, but not too many mathematicians (the latter being immune to such surprises in science—at least from outside mathematics proper).

Mathematical economics has been enriched by a series of papers of Smale [123–128], from the study of static modes of equilibria to the dynamics of adjustment processes and the convergence of the economic state towards equilibria (such as defined by Pareto). These investigations lead to polysystems that are important generalizations of Morse-Smale systems [122]. Another topic concerns the optimization of the growth of a controlled economy, and the corresponding mathematics of Hamiltonian differential systems [110].

II. Diffeomorphisms and Foliations

Perhaps the most impressive recent contributions to topics of this appendix, at least within pure mathematics, have come from differential topology, where it impinges on differentiable dynamics.

For example, the results of P. Schweitzer [107] give a C^1-flow counterexample to the Seifert conjecture on S^3 (see page 4). Also consider the contrary result of A. Fathi and

M. Hermann [25] on the existence of a C^∞-diffeomorphism that generates a minimal discrete dynamical system on S^3. The principal problem in this area lies between these two results: *Is there a C^∞-flow on S^3 that is minimal; must every noncritical C^∞-flow on S^3 have a periodic orbit*? (See STOP PRESS, end of Appendix.)

It is known [111] that no diffeomorphism of S^3 (or higher dimensional S^n) can be hyperbolic in the sense of Anosov, but the question for flows, or even of the minimality of flows on spheres is a major unresolved problem.

From a different viewpoint we can consider actions of the circle group S^1 on S^3—so that each orbit is periodic, and also the entire flow has a common period. The easiest example is the Hopf foliation of S^3 [39]. It has been proven that each noncritical flow on S^3, in which each orbit is periodic, is equivalent to such a periodic group action [24]. The difficulty is to show that there is a longest period among all the periodic orbits on S^3; while this at first glance seems highly plausible, ingenious counterexamples [131] on compact manifolds of higher dimensions illuminate the profundity of the proof.

Group actions of $\mathbf{R}^2$, and other Lie groups, on spheres yield generalizations [18, 69, 96] of the classical Poincaré-Bendixson theorem on the existence of closed orbits. These studies are intermediate between the classical theories of differential systems and the geometric theories of foliations, in which much progress has been made. For instance, there is a foliation of codimension 1 on each compact manifold with Euler characteristic zero [136].

Besides the Poincaré-Bendixson theorem for flows on the sphere S^2, the other famous result of differential topology for flows on surfaces is the Denjoy Theorem for the torus T^2. That is, the only minimal sets for a smooth flow on T^2 are either periodic orbits or else the entire torus surface. Several new types of interesting flows on higher dimensional tori have been constructed, including new minimal flows [40, 114]. Thus various kinds of nonequivalent minimal flows exist on an n-torus. (Related remark: an Anosov diffeomorphism of a torus or any nilmanifold must necessarily be topologically one of the standard kinds related to the algebraic structure of the phase-space [65, 66].)

The Denjoy analysis for C^∞-flows on T^2 rests on a study of diffeomorphisms of the circle S^1. Each such diffeomorphism (orientable and with irrational rotation number) is topologically equivalent to an irrational rotation of S^1. A major new result of M. Hermann [35] shows that, for a certain dense set of rotation numbers, there is a C^∞-diffeomorphic equivalence with the irrational rotation.

Finally we mention an interesting result of J. Harrison that certain C^k-diffeomorphisms of T^2 have topological invariants that depend on the order k of differentiability [34]. Hence some C^2-diffeomorphisms of tori are not even topologically equivalent to any C^∞-diffeomorphism. Similar results hold for flows on higher dimensional manifolds.

III. General Theory—Dissipative Dynamics

1. Hyperbolic Invariant Sets. Structures are such stuff as dynamics are made of. So we first must study the structure of invariant sets, and then the structure of a whole dynamical system—primarily from the viewpoint of stability. Thus we consider a dynamical system or vector field v on a C^∞-differentiable n-manifold M. For simplicity, as in the original monograph, we take M compact and denote by $\mathfrak{V}(M)$ the space of C^1-vector fields on M. In this terminology, employed in this monograph, we shall indicate some of the newer results of differentiable dynamics–emphasizing the nature and direction of research and de-emphasizing any complete recitation of technical hypotheses.

Near a critical point or periodic orbit of a dynamical system $v \in \mathfrak{V}(M)$, which is of hyperbolic type, the corresponding linear structure is qualitatively correct. That is, there exists a topological linearization of the dynamical system v, as is reported earlier. It is now known that there likewise exists a valid linearization near any invariant compact submanifold N, provided there is a hyperbolic structure for the induced flow on the normal bundle and furthermore this normal component of the flow dominates the tangential flow on N (which is always the case for equicontinuous flows on tori) [36, 37]. In the extreme case where the normal component of the flow tends towards N, so the invariant manifold N is (future) asymptotically stable, then N is called an attractor both for continuous and discrete flows.

During the past several years there has been much interest in the general study of attractors–compact invariant sets N (usually with at least one dense orbit) that are future asymptotically stable. Sometimes N is not a manifold, and the induced flow on N is chaotic (say, topologically mixing); in which case N is called a "strange attractor" (see V). Expanding attractors have special homological properties; for instance, they bear Čech cycles [132, 143].

The property of hyperbolicity is closely related to the dominant theme of qualitative dynamics–namely, structural stability. For example, hyperbolic Anosov flows display an appropriate hyperbolic decomposition of the tangent bundle of M, and a basic theorem asserts (see page 36) Hyp $\subset$ SS. In the case of conservative Anosov flows the nonwandering set Ω fills the entire manifold M, since the set (Per) of periodic orbits is dense in M. The corresponding result is not known for general Anosov diffeomorphisms but it is definitely false for Anosov flows (even though Per is an infinite set dense in Ω) [27].

A very significant generalization of the concept of Anosov hyperbolic flows on M are the flows satisfying Axiom A:

A) $\overline{\text{Per}} = \Omega$, *and* Ω *is everywhere hyperbolic.* The requirement of hyperbolicity in Axiom A is that at each point P of the compact set Ω the tangent space $T_P M$ decomposes into a direct sum of a (0 or 1 dimensional) space along the flow and then a dilating and a contracting subspace–satisfying the exponential bounds and conditions indicated on page 31. In particular, note that each critical point and periodic orbit of an Axiom A system is necessarily hyperbolic.

It is also possible to generalize the analysis of Anosov systems by expressing the demand for hyperbolicity in terms of the Liapunov-Perron characteristic exponents–say, assume these exponents are bounded away from zero on M, or just on Ω, with some attention to appropriate uniformity demands.

Very roughly speaking the Anosov hyperbolic flows on M are related to conservative dynamics, whereas Axiom A systems refer to dissipative dynamics. For instance, Axiom A systems include all Morse-Smale systems on M, that is,

$$\text{MS} \subset \text{Ax. A.}$$

2. Structural Stability. Both classes SS and Ax. A are plausible models for a general theory of "physical dynamics with dissipative friction". It is a most important problem of differentiable dynamics to clarify the interrelations of these two classes of $\mathfrak{V}(M)$. For this purpose we supplement Axiom A by another Axiom B:

B) *Strong transversality holds for all attracting and repelling manifolds for* Ω (*which is hyperbolic in accord with Axiom* A).

In order to clarify Axiom B we remark that each point $P \in M$ has a trajectory that approaches (say, as $t \longrightarrow +\infty$) the trajectory through some point $Q \in \Omega$. This theorem on the existence of an "asymptotic phase" of P is proved by using the local product structure near each basic set (transitivity component) of Ω. Thus the point $Q \in \Omega$ specifies an attracting and a repelling submanifold of M, and hence we obtain two foliation-like decompositions of M. The demand of Axiom B is that any intersection in M of such an attracting and a repelling submanifold (for all pairs of points of Ω) must be transversal. (For diffeomorphisms this implies that the expanding and contracting subspaces of each point $x \in M$ must span the entire tangent space at x).

An important theorem asserts that Axioms (A and B) imply structural stability, in fact a stronger conclusion of absolute SS. Here the class (absolute SS) consists of systems $v \in$ SS with the further requirement that the conjugating homeomorphism Ψ of M, carrying the sensed trajectories of v to its neighbor $(v + \delta v) \in \mathfrak{V}(M)$, differs from the identity by $\delta\Psi$; and moreover $\delta\Psi$ has a Lipschitz-bound in terms of δv. The best result in this direction is that the dynamical systems satisfying both Axioms A and B are precisely the systems that are absolutely structurally stable [97, 100, 101]:

$$\text{Ax. (A and B)} = \text{absolute SS}.$$

An outstanding open problem is whether the above equality also holds for the class SS (in other words, is SS = absolute SS?).

Besides the problem of characterizing the class SS by the behavior of the nonwandering set Ω, we can investigate the extent to which SS is (or fails to be) generic in $\mathfrak{V}(M)$. For example it is known (see monograph, page 36) that SS is not C^1-dense in $\mathfrak{V}(M)$. However new results prove that SS is C^0-dense in the set $\mathfrak{V}(M)$. In order to obtain this result we must define some subclass of SS that is easy to analyse—wider than MS, yet still rather explicit. (In passing we note that MS is not C^0-dense in $\mathfrak{V}(M)$—although it is known [10, 79] that each homotopy class of noncritical vector fields in $\mathfrak{V}(M)$ does contain a Morse-Smale system, for $n \geqslant 4$.)

Recall that a Morse-Smale system $v \in$ MS satisfies the axioms (page 28): Ω = Per, which consists of finitely many critical points and periodic orbits, each hyperbolic; and all attracting and repelling manifolds have transversal intersections. To enlarge MS appropriately we permit a further finite collection of compact invariant 1 dimensional sets in Ω, known as "horseshoes" (or simply "shoes"—see reference on page 37—which each contain homoclinic points and thus infinitely many periodic orbits). Then a Smale system has a nonwandering set Ω consisting of finitely many isolated critical points, periodic orbits, and shoes—all satisfying the usual types of hyperbolicity, transversality, and so noncyclicity hypotheses. Thus

Smale systems satisfy Axioms (A and B) and hence belong to SS. Yet it is known [26] that Smale systems are C^0-dense in $\mho(M)$. As a slight simplification of the class of Smale systems, Zeeman uses only planar saddle shoes (The Goodie Two Shoes Construction) [147].

3. Bifurcation. During the past decade there have been deep studies of the bifurcation set, that is, the closed subset of $\mho(M)$ that is the complement of SS. Some of these studies refer to very general properties of bifurcation, and others to very detailed analyses of particular types of bifurcation. In this section we comment mainly on the general nature of bifurcation, and in area V there is some mention of special and chaotic bifurcations.

Consider a generic arc $\zeta(s)$ in the group Diff(M) of all diffeomorphisms of the manifold M. It has been shown that $\zeta(s)$ fails to be Kupka-Smale [130] at only a countable number of points, and at these points the properties of hyperbolicity or transversability collapse in a limited and elementary manner. There is also a deep study [85] of a generic arc $\xi(s)$ in $\mho(M)$ approaching the bifurcation set from the open set MS; but the analysis is not complete if the intricacies of the limiting behavior of $\xi(s)$ near the boundary of MS are allowed the most general complexities.

In a somewhat different vein, it is possible to join two components of MS by an arc in $\mho(M)$ with only a finite number of bifurcation points—each of which corresponds to a Kupka-Smale (or quasi-KS) system [87]. Using these ideas of continuous deformations of MS systems, Peixoto [92] has remarked that the Morse-Smale inequalities (see page 30) can be improved:

$$M_0 \geqslant b_0,$$

$$M_1 - M_0 \geqslant b_1 - b_0,$$

$$\vdots$$

$$\sum_{j=0}^{n} (-1)^{n-j} M_j = \sum_{j=0}^{n} (-1)^{n-j} b_j,$$

where b_j is the Betti number specified as the dimension of the real j-homology group $H_j(M, \mathbf{R})$.

In addition to these global bifurcation results there is a profound analysis of generic bifurcations of critical points, starting with Hopf bifurcation for one parameter, and then dealing with several parameters, and catastrophe singularities. We refer back to these topics later in area V.

4. Axiom A. The induced dynamics on the nonwandering set Ω is reasonably understandable for dynamical systems satisfying the hyperbolicity demand of Axiom A. Hence there has developed an entire research group engaged in a profound analysis of the structure and nature of attractors in Axiom A systems [12, 13, 14]. Some of the major results here involve ergodic theory, especially concepts of entropy and mixing.

For technical reasons we consider Axiom A flows of differentiability class C^2 on a compact C^∞-manifold M. Then for almost all initial points (with regard to the obvious null sets of M), the corresponding trajectory tends to an attractor as $t \longrightarrow +\infty$. The basin of an attractor Λ is the set of all points that tend to the specified attractor Λ as $t \longrightarrow +\infty$. Moreover, almost all points of M belong to the basin of some attractor.

Under these circumstances we can state the fundamental ergodic theorem as it applies to a C^2-vector field v, satisfying Axiom A on M [15, 104]:

Let Λ be an attractor for the C^2-vector field $v \in$ Ax. A, with flow $t \longrightarrow x(t)$ on M. Then there exists a unique invariant probability measure μ, with support Λ, such that :

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \Phi(x(t))\,dt = \int_\Lambda \Phi(y)\mu(dy),$$

for almost all initial points $x(0)$ in the basin of Λ; where $\Phi(y)$ is any real continuous function on this basin. Moreover μ is ergodic under the flow of v on Λ.

Note that $x(0)$ can be any point of the basin of Λ, excluding a (Lebesgue) null set. Also the support of μ is Λ, so that μ may well be singular with regard to Lebesgue measure on M. While μ is necessarily ergodic for v on Λ (that is, the only invariant subsets have measure 0 or 1 on Λ), the flow of v might not be μ-mixing. But the induced map on some appropriate cross-section of Λ is measure-equivalent to a subshift of finite type, and hence equivalent to a Bernoulli shift.

The topological entropy h of a continuous map χ of M into itself is a numerical indicator of the tendency of orbits to separate apart. To define h consider k-segments of orbits (that is $x, \chi x, \chi^2 x, \ldots, \chi^{k-1}x$ for $x \in M$) and let $h(k, \epsilon)$ be the least number of points in M whose k-segments are ϵ-dense—each k-segment lies uniformly within a distance $\epsilon > 0$ of some one of a given collection of k-segments. Then the entropy of χ is defined:

$$h(\chi) = \lim_{\epsilon\to 0}\ \limsup_{k\to\infty}\left[\frac{1}{k}\log h(k, \epsilon)\right].$$

A fundamental problem concerns the relation of the entropy h to the induced homology map χ_* on M. For Axiom A diffeomorphisms $h \geqslant \log|\lambda|$, where $|\lambda|$ is the largest among the eigenvalues of the homology map χ_* (in all dimensions) [11, 13]. In fact for Anosov maps, equality holds provided certain orientability hypotheses are valid. For general homeomorphisms the above inequality fails [93], although it is valid for $|\lambda_1|$ largest among the eigenvalues for the first homology group [67, 68].

If χ is a topological map of a compact n-manifold M onto itself, preserving a measure μ, then the entropy (with respect to μ) does not exceed the topological entropy $h(\chi)$. In fact,

$$h(\chi) = \sup h(\mu)$$

where μ runs over all invariant measures of χ. For the case where χ is a C^2-diffeomorphism satisfying Axiom A, we can further assert that there exists an invariant measure $\hat{\mu}$ such that $h(\chi) = h(\hat{\mu})$—and moreover $\hat{\mu}$ can be described by the density of the various periodic points of χ [11, 13, 15, 103]. Also the ζ-function that counts the periodic orbits is then a rational function [64].

In terms of the Liapunov exponents of the C^2 diffeomorphism χ we obtain [S29]

$$h(\mu) \leqslant \int_M \sum \text{(positive Liapunov exponents)}\, d\mu.$$

If the Liapunov exponents of χ are nowhere zero on M, and there is an invariant ergodic, nonatomic measure on M, then there exist transverse homoclinic points for χ—in fact, their support contains the support of μ. In this case the entropy $h(\chi) > 0$, see [48, 49]. A partial converse holds on surfaces where $h(\chi) > 0$ implies the existence of homoclinic points.

IV. General Theory–Conservative Dynamics

The simple harmonic oscillator fails to be structurally stable in $\mathbf{R}^2$ if perturbations are allowed within the class of all C^1-vector fields in the plane; yet within the restricted class of conservative systems (see page 8) $\ddot{x} + g(x) = 0$ the structure of a center is maintained. However, note that the 2 dimensional linear oscillator (page 3) $\ddot{x}^1 + k^2x^1 = 0$ and $\ddot{x}^2 + l^2x^2 = 0$ (position (x^1, x^2)), which can be analysed in the (x^1, x^2, y_1, y_2)-space $\mathbf{R}^4$ by

$$\dot{x}^1 = \partial H/\partial y_1, \quad \dot{y}_1 = -\partial H/\partial x^1 \quad \text{and} \quad \dot{x}^2 = \partial H/\partial y_2, \quad \dot{y}_2 = -\partial H/\partial x^2$$

with Hamiltonian energy function

$$H(x^1, x^2, y_1, y_2) = \frac{(kx^1)^2 + (lx^2)^2}{2} + \frac{(y_1)^2 + (y_2)^2}{2},$$

fails to be structurally stable in $\mathbf{R}^4$—even within the class of Hamiltonian conservative systems. This is evident since the flow for this oscillator in $\mathbf{R}^4$ has periodic orbits if and only if k/l is rational.

1. Global Hamiltonian Systems on Symplectic Manifolds. These elementary considerations show that the general theories of structural stability and of generic behavior must be profoundly modified if they are to contribute to the study of conservative dynamics, say for Hamiltonian systems

$$\dot{x}^i = \partial H/\partial y_i, \qquad \dot{y}_i = -\partial H/\partial x^i, \qquad i = 1, \ldots, n$$

for given Hamiltonian functions $H(x^1, \ldots, x^n, y_1, \ldots, y_n)$ in class C^2 (usually in C^∞) in the (x, y)-space $\mathbf{R}^{2n}$. Furthermore, in order to construct a global theory of Hamiltonian dynamics on a C^∞-differentiable $2n$-manifold M we require the existence of a distinguished atlas of canonical (or symplectic) local charts (x, y) that are interrelated by canonical coordinate transformations—and hence maintain the format of the given Hamiltonian vector field. That is, the corresponding Jacobian matrix T of the coordinate transformation must be symplectic at each point:

$$TJT' = J = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

Then M, with the symplectic atlas of canonical charts, specifies a symplectic manifold. The globally defined nonsingular 2-form $\Omega = \Sigma_{i=1}^n dx^i \wedge dy_i$ (in each canonical chart) is the symplectic form on M, and Ω specifies a duality between tangent vectors and covectors at each point of M. Each real C^2-function H on M defines a gradient covector dH, and by Ω-duality, a tangent vector field $dH^{\#}$ called the Hamiltonian vector field for the Hamiltonian function H. In each canonical chart (x, y) the components of $dH^{\#}$ are $(\partial H/\partial y_i, -\partial H/\partial x^i)$ and hence the Hamiltonian differential system has the classical format. If M is compact, or if M is the cotangent bundle of a compact n-manifold, then each Hamiltonian vector field defines a complete flow on Ω. We refer to standard texts and recent expositions [1, 72, 83] for the further discussion of the differential topology of symplectic manifolds, and the related structures of classical conservative dynamics–including such concepts as Lagrangian submanifolds, integrals of motion, and involutory systems.

2. Liapunov Periodic Orbits and Generalizations. Near a critical point, say at the origin of canonical chart (x, y), the Hamiltonian function can be written

$$H = \frac{1}{2}(x, y)S\begin{pmatrix} x \\ y \end{pmatrix} + \cdots \quad \text{for } S = S',$$

and the Hamiltonian differential system becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = JS\begin{pmatrix} x \\ y \end{pmatrix} + \cdots .$$

The critical point is called elliptic in case every eigenvalue of the Hamiltonian matrix JS is pure-imaginary (the linearization displays neutral stability). In particular, this occurs when S is positive definite, so H achieves a strict local minimum at the origin.

Nearly a century ago Liapunov proved that there exist n families of periodic orbits near a generic elliptic critical point (eigenvalues suitably independent over rationals), corresponding to the n normal modes of vibration of the linearized system. Also each of the families is parametrized by the energy level of the vibration. Recent research of Weinstein [141, 142] proves the existence of such "Liapunov periodic orbits" without the imposition of special irrationality conditions, near a minimum of H where $S > 0$. A generalization of the method of Liapunov can be used to prove the existence of periodic orbits near to those small tori arising in the linearized system [139].

3. KAM-Theory. Some of the most difficult and profound contributions to the theory of ordinary differential equations during the past generation have dealt with the local behavior of Hamiltonian systems near a critical point, or a periodic orbit, or even a higher dimensional invariant torus. More specifically, in the elliptic case the methods of Kolmogoroff-Arnold-Moser (KAM-theory) overcome the difficulty of "small-divisors" to prove the convergence of suitable approximations demonstrating the existence of invariant tori of various dimensions [9, 50, 82].

For example, consider a periodic orbit σ for a Hamiltonian system $dH^{\#}$ in $\mathbf{R}^4$ (so $n = 2$) and restrict attention to the corresponding energy hypersurface $H =$ constant (often this is S^3 in $\mathbf{R}^4$). Take a 2-section Σ transversal to σ in the energy hypersurface and consider the Poincaré return map Π of Σ. In the elliptic case both eigenvalues of Π (or its linearization near σ) are of modulus 1, that is, the nontrivial characteristic multipliers of σ are $\mu = e^{2\pi if}$ and $\mu^{-1} = e^{-2\pi if}$ (for real frequency f, mod 1). Now assume that the frequency f is suitably irrational, and that the map Π is strictly nonlinear so that the "angular twist" of Σ varies with the distance out from σ. Then the KAM-theory [83, 84] guarantees the existence of a simple closed curve $\mathcal{C}$, encircling the point $\sigma \cap \Sigma$ in the 2-section Σ, and invariant under the Poincaré map Π. In this case $\mathcal{C}$ generates an invariant 2-torus surface $T^2(\mathcal{C})$ that encloses a tubular neighborhood of the periodic orbit σ, so σ is necessarily Liapunov stable. A further result is that the induced flow on $T^2(\mathcal{C})$ is almost periodic with an irrational rotation number.

The existence of the invariant 2-torus $T^2(\mathcal{C})$, and the enclosed tubular neighborhood of σ, shows that the Hamiltonian flow fails to be ergodic on the given energy hypersurface. In higher dimensions, say $n \geqslant 3$, the corresponding invariant n-tori do not topologically separate the energy $(2n - 1)$-hypersurface and Liapunov stability for σ can hold or fail, depending on the nature of the higher order terms of Π, see [82, 84]. However, ergodicity must fail since there exists a sufficient collection of invariant n-tori to fill a subset of positive measure in the given hypersurface.

The KAM-theory was not discussed previously in the original monograph, partly because of the ignorance of the author, and partly because the theory was local and so did not fall within the province of the global theory of differentiable dynamics. However, during the past decade many of the local KAM results have been reformulated and modified so as to fit into the developing theory of global conservative dynamics. (Also this author has now studied these results—to alleviate ignorance—and has even made some contributions in these subjects, as indicated later.)

4. Generic Hamiltonian Systems. The first studies of the global theory of conservative dynamics used the same techniques (adequately refined) as for the general theory of differentiable dynamics on manifolds. In this way the corresponding closing-lemma (see page 37) was proved, and important generic classes of Hamiltonian systems were specified in terms of the nature of their critical points and periodic orbits [94, 98, 99]. For instance, a generic class of Hamiltonian systems has all (excepting a possible countable subset) periodic orbits that are nondegenerate (nontrivial characteristic exponents different from 1) [98].

Using these generic properties of Hamiltonian systems, together with KAM-theory, Markus and Meyer have obtained the results indicated in the title of their Memoir [72], *Generic Hamiltonian Dynamical Systems are neither Integrable nor Ergodic.* Continuing the direction of their researches in Hamiltonian dynamics, the same two authors have recently published a substantial paper [73] showing that generic Hamiltonians possess complicated sequences of long periodic orbits, with accumulations towards solenoidal minimal sets (that is, 1 dimensional non-locally-connected continua). The main theorem in this work refers to the Baire space $\mathcal{H}^k$ of all Hamiltonian dynamical systems, of class C^k $(4 \leqslant k \leqslant \infty)$ on a compact symplectic C^∞-manifold M of dimension $2n \geqslant 4$.

Let $\mathcal{H}^k$ be the space of Hamiltonian dynamical systems on the compact symplectic manifold M. Then there exists a generic set $\mathcal{M}_\Sigma \subset \mathcal{H}^k$ such that: for each Hamiltonian *system* $dH^{\#} \in M_{\Sigma}$, *and for each solenoid* Σ_a, *there exists a minimal set, for the flow of* $dH^{\#}$, *that is homeomorphic to* Σ_a.

V. Chaos, Catastrophe, and Multivalued Trajectories

The complexity of various physical and geometrical phenomena tends to cast doubt on the universal validity of the principle of determinism—that the future states of a system necessarily evolve from the present state, according to fixed dynamical laws. Important examples of such complexity, with corresponding uncertainty, include: (i) molecular chaos with ergodic mixing, as interpreted by geometric models of geodesic flows over manifolds of negative curvature; atmospheric turbulence as interpreted by the Lorenz equations; (ii) jump-phenomena in nonlinear vibration analysis; (iii) fluctuations in electronic circuits; branching of controlled spaceship trajectories; and (iv) the spreading of biological epidemics. In these examples the concept of the "present state" appears unclear and indefinite, since the future evolution might branch in diverse ways depending on hidden parameters, or even on factors of uncertainty and interference.

Instead of ignoring the principle of determinism, a more conservative philosophical approach would be to broaden the concept of "state" to cope with these branching evolutionary possibilities. There is a well-established tradition, over the past half-century, in dealing with such problems. Consider, for instance, the use of hidden parameters in the thermodynamics of imperfect gases; the probabilistic basis of quantum mechanical uncertainties and of Brownian motion, with a resulting vast literature on stochastic processes; and the historical description of elastic media in studies of hysteresis.

We next list some recent attempts to resolve similar problems and to incorporate the apparent "weakening of determinism" into the framework of the strict determinism of the theory of dynamical systems.

First, the geometric complexities can be tackled head-on, say as in the horseshoe construction for a homoclinic point or some other intricate flows that describe ergodic or mixing phenomena. Note that in dynamical systems with such chaotic behavior, the trajectories are highly sensitive to slight changes in the initial data. That is, very small changes in the initial data will yield trajectories with wildly diverse asymptotic behavior. This gives a possible mathematical interpretation of physical phenomena such as fluid turbulence. In order to bypass the details of the complicated geometry, one can turn to probabilistic interpretations—for instance, ergodic theory itself provides such a simplification in that it refers to mean asymptotic behavior of trajectories.

Second, the dynamics might contain a finite number of hidden variables or parameters that are adjusted from time-to-time so as to produce qualitative changes in the dynamical system—for instance, bifurcations or even more startling catastrophic discontinuities of behavior.

These first two approaches fit rather closely into the classical theory of deterministic dynamical systems having a finite number of degrees of freedom. The next two methods of modification enlarge the notion of "state", or the basic initial data required to determine the future, to an element in an infinite dimensional function space. For instance, the system may be affected at each instant by some external influence—either probabilistic (as for stochastic differential equations), or by willful control inputs. In both these cases the dynamical system is no longer finitistic in nature, since more than a choice of an initial point on a finite dimensional manifold is required to specify the future evolution. That is, an infinite set of data must be introduced, a stochastic process, or a choice of a control function—as supplementary data input.

Finally, the state can incorporate past historical data, as for a differential-delay equation (differential equation with retarded arguments)—or more generally, a differential-functional system. Of course, such differential-functional equations can further be generalized to include control elements—even stochastic control. (Why not? Nothing in mathematics is so ludicrous that it has not already been taken seriously.)

1. **Chaos and Ergodic Mixing.** Classical mechanical problems that deal with chaotic dynamics in very high dimensions (like molecular statistical mechanics) are often treated by the methods of geometric ergodic theory. Some of these dynamical systems may be described as Anosov systems arising as geodesic flows over compact manifolds of negative curvature [4, 117, 118]. It would take us too far afield to report on the many developments within ergodic theory proper, so we restrict attention to a few remarks especially pertinent to differentiable dynamics. For instance, the general study of the ergodic theory of Anosov flows, with corresponding mixing properties, is well known [4, 90, 140]. It should also be remarked that every compact n-manifold (for $n \geqslant 3$) admits smooth ergodic (that is, metrically transitive with respect to a smooth invariant measure) differentiable flows [6].

Recent efforts to explain the chaotic behavior of some deterministic dynamical systems examine very special differential equations, having trajectories delicately sensitive to the initial conditions. For example, consider the Lorenz differential equations in $\mathbf{R}^3$:

$$\dot{x} = \sigma y - \sigma x, \qquad \dot{y} = -xz + rx - y, \qquad \dot{z} = xy - bz$$

for positive parameters (σ = Prandtl number, r = Rayleigh number, $b > 0$). The Lorenz equations [54] arose in a simplified model of atmospheric turbulence. Computer studies of the Lorenz system (originally using the parameter values $\sigma = 10$, $r = 28$, $b = 8/3$) indicated the existence of a strange attractor Λ (for the Poincaré map on the plane $z = 27$). It seems likely, although not proven mathematically, that the discrete dynamics on Λ can be related to the study of a map χ of the linear interval $[-1, 1]$ into itself. One model for the Lorenz attractor is

$$\chi\colon x \mapsto \begin{cases} 2x + 1 & \text{on } -1 \leqslant x \leqslant 0, \\ 2x - 1 & \text{on } 0 < x \leqslant 1, \end{cases}$$

but the ergodic properties of the dynamics seem to depend only on the qualitative form of χ. For such maps χ there is always an absolutely continuous invariant measure [91, 145].

2. **Bifurcation, Catastrophe.** Great effort has been expended on these various models for the Lorenz attractor, and for other types of chaotic dynamics that can be represented by various maps of linear intervals. For instance, the dynamics of the horseshoe invariant set are closely related to those of the map of the segment $0 \leqslant x \leqslant 1$ described by

$$\chi_a: x \mapsto ax(1-x) \quad \text{with parameter } 0 < a \leqslant 4.$$

Each of these maps χ_a satisfies Axiom A and has a unique attracting periodic orbit, at least for values of the parameter a outside some nowhere dense set (related results hold for all qualitatively similar maps [32, 42, 52, 78]). When $0 < a < 1$ the map χ_a is a contraction back toward the fixed point at $x = 0$. But as the parameter a increases, there appear more and more unstable periodic points, through ever more complicated bifurcations—eventually leading to a cascade of bifurcations and chaotic behavior.

In contrast to the above extremely complicated pattern of bifurcations, the classical studies of bifurcations have been more elementary and have been used to describe the appearance of periodic orbits of rather special dynamical systems—as some parameter passes through a critical bifurcation value.

The most famous treatment of bifurcations of periodic orbits (after the classic studies of Poincaré on celestial mechanics during the last century) is the Hopf bifurcation [38, 61]. The Hopf bifurcation of a limit cycle from a critical point applies to a differential system in the plane $\mathbf{R}^2$, as the scalar parameter changes the nature of the equilibrium from stable to unstable. Using the center manifold theorem [1] this result can be applied to systems in $\mathbf{R}^n$—but the bifurcation activity is essentially in a 2-manifold and still depends on a single scalar parameter.

Much more complicated bifurcation structures arise when two or more parameters enter the picture [133]. Catastrophe theory offers the most detailed and complete analysis of bifurcations of the structure of the critical points of a real differentiable function, depending on five or less real parameters. Such bifurcations have been classified and modelled by a finite list of qualitative geometric forms known as the elementary catastrophes [28, 135]. Hence we can study the bifurcation of the critical points of a potential function (for a gradient differential system), or for a Hamiltonian function (for the corresponding Hamiltonian differential system). However, some care must be exercised to distinguish between the qualitative behavior of the potential function (or Hamiltonian function) and that of the corresponding dynamical system [29], with respect to the bifurcations arising from the changing parameter values.

The theory of catastrophes, as expounded profoundly by R. Thom [135] is a geometric theory of metaphysics. Further studies [149] have clarified the mathematical standing of the elementary catastrophes, and have developed the theory from the viewpoint of global dynamics. While the theory of catastrophes might, with suitable caution, serve as a basis for the organization of certain phenomenological approaches to the biological and behavioral sciences [28, 148], the method fits more traditionally into the theory of nonlinear mechanics, say nonlinear vibrations and elasticity [28]. It seems plausible to expect that many discontinuities and jump-phenomena, say for the forced Duffing oscillator, can be clarified and explored by referring to the geometry of the elementary catastrophes [19].

3. Stochastic Dynamics, Control Dynamics, and Polysystems. We shall define a stochastic differential system on a differentiable n-manifold M, in terms of a deterministic flow (along a given C^1-vector field $X_0(x)$) and a stochastic disturbance of white noise (indicated by $\dot{w}(t)$, where $w(t)$ is the real scalar Wiener process of Brownian motion—as defined on some probability space Ω) acting via given tangent C^1-vector fields $X_1, \ldots, X_m$ on M. In each local chart (x) we write the corresponding stochastic differential system

$$\dot{x} = X_0(x) + \dot{w}^1(t)X_1(x) + \cdots + \dot{w}^m(t)X_m(x).$$

In order that this stochastic differential system should be globally defined on M (with respect to the usual Îto calculus in coordinate transformations), the integral stochastic process $x(t)$, from given initial state $x_0 \in M$, should be interpreted in the sense of Stratonovich [20].

The general theory of such global stochastic dynamical systems has been developed in great generality in the work of Eells and Elworthy [S14]. In particular Brownian motion over a general Riemannian n-manifold can thus be defined by a stochastic differential system in the frame bundle, where $X_0 \equiv 0$ (no drift) and the vectors $X_1, \ldots, X_n$ are the natural liftings, via the Christoffel connection, of frames on the base manifold to horizontal vectors in the frame bundle. In the corresponding theory of relativistic Brownian motion over a Lorentz manifold, the appropriate stochastic system in the frame bundle keeps X_0 as the lift of the time-like vector of each Lorentz orthonormal frame, and the remaining vectors $X_1, \ldots, X_{n-1}$ are the lifts of the space-like vectors of the base frame of the Lorentz manifold [71].

In the general theory of stochastic dynamical systems on M the solution $x(t)$, initiating at x_0, is a stochastic process that lies almost surely on a certain submanifold through x_0—namely the same submanifold that contains the attainable (or reachable) set for the analogous control dynamical system on M [44, 53]:

$$\dot{x} = X_0(x) + u^1(t)X_1(x) + \cdots + u^m(t)X_m(x).$$

Here $u^1(t), \ldots, u^m(t)$ are real scalar controllers of some specified class, say piecewise constant, and the vector fields $X_0, X_1, \ldots, X_m$ are just as before. In the homogeneous case, where $X_0 \equiv 0$ on M, the attainable set is precisely the submanifold of M obtained by integrating the Lie algebra generated by $\{X_1, \ldots, X_m\}$ (in the analytic case, otherwise some further extension is required [53]).

For the case of a group-invariant control system—that is, M is a Lie group G (say, matrix subgroup of $\mathrm{GL}(n, \mathbf{R})$), with right-invariant control vector fields, we have the matrix control dynamics

$$\dot{X} = A_0X + u^1(t)A_1X + \cdots + u^m(t)A_mX.$$

Here the real $n \times n$ matrices $A_0, A_1, \ldots, A_m$ each belong to the Lie algebra g of G, so each vector field $A_0X, A_1X, \ldots, A_mX$ is right-invariant on G. Thus each solution trajectory $X(t)$ initiating from $X_0 \in G$ must lie always within the matrix group G. In the homogeneous case, where $A_0 \equiv 0$, the attainable set from E is precisely the Lie subgroup with Lie algebra generated by the matrices $\{A_1, \ldots, A_m\}$. Hence the homogeneous system is controllable in G if and only if the matrices $A_1, \ldots, A_m$ generate the Lie algebra g of G, [16, 43]. For the nonhomogeneous case further hypotheses are required to guarantee controllability on G: for instance, G is compact and $A_0, A_1, \ldots, A_m$ generate its Lie algebra [43].

Another approach to the study of control dynamics, on a differentiable n-manifold M, is via the terminology of polysystems. To illustrate this interrelation consider a control problem on M defined by a control differential system $\dot{x} = f(x, u)$ (in each local chart (x) on M); that is, f is a differentiable cross-section map of $M \times \mathbf{R}^m$ into the tangent bundle of M. Here the control parameter u denotes the various admissible controllers, say piecewise constant $u(t)$ in $\mathbf{R}^m$, each producing a solution trajectory $x(t)$ from a specified initial point x_0 in M. That is, $x(t)$ describes a piecewise integral curve of the various vector fields $f(x, u_0)$, for various constant values of $u_0 \in \mathbf{R}^m$. Hence we consider the family $\mathcal{D} = \{X_u(x)\}$ of vector fields on M, where $X_u(x) = f(x, u)$ for each fixed $u \in \mathbf{R}^m$. Then the trajectory $x(t)$ of the control system, for a given piecewise constant controller $u(t)$ on $0 \leqslant t \leqslant T$ and a given initial point $x_0 \in M$, is recognized as a trajectory of the polysystem $\mathcal{D}$. The study of polysystems, or multivalued differential systems on M, then relates immediately to control dynamics.

The controllability of nonlinear dynamical systems on a compact differentiable n-manifold M has been successfully phrased in terms of polysystems [53]. In this conceptual framework an important theorem asserts that generic 2-polysystems are completely controllable on M (provided we use symmetric or reversible polysystems—that is, effectively permit both past and future trajectories). For conservative systems only future-control need be employed, and in this case generic 2-polysystems are controllable on M, [53].

The prior methods of geometric analysis have applications to engineering control problems [16], and also to the stability of economic markets where continuous adjustments towards a Pareto equilibrium are pursued [122].

4. Differential-Functional Systems. Even the simplest differential-delay equation

$$\dot{x}(t) = ax(t) + bx(t-1)$$

for real constant coefficients a, b and real scalar solution $x(t)$ on $t \geqslant 0$, leads to interesting and novel problems. For instance, where are the complex roots λ of the characteristic equation $\lambda = a - be^{-\lambda}$, and for which values of a and b will each solution $x(t)$ decay eventually towards zero [22, 33]? Of course, appropriate initial data for a solution $x(t)$ on $t \geqslant 0$ consist of a segment of a real continuous function $x_0(s)$ on $-1 \leqslant s \leqslant 0$; and then the solution $x(t)$ is uniquely defined on $t \geqslant 0$ by the method of step-wise integration.

More general linear systems for $x \in \mathbf{R}^n$ are defined by

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t-h_1) + \cdots + A_m(t)x(t-h_m) + B(t)$$

for positive (usually constant) delays $h_1, \ldots, h_m$, and continuous coefficient matrices $A_0(t), A_1(t), \ldots, A_m(t)$, and column vector $B(t)$. Even more general hereditary systems correspond to a continuum of delays, say

$$\dot{x}(t) = \int_{-\infty}^{0} A(t, s)x(t+s)\,d\mu(s) + B(t),$$

where μ is a suitable measure on $\mathbf{R}$—however, we shall refer only to the case of a finite number of delays so the time-horizon $h = \max\{h_1, h_2, \ldots, h_m\}$ is finite. In this case the delays can be incorporated into the algebraic structure of the coefficients by introducing the shift operators [63, 81]:

$$\sigma_j\colon x(t) \mapsto x(t - h_j), \qquad j = 1, \ldots, m.$$

(Note that the σ_j commute as elements of the ring of continuous linear operators of, say, the linear space $L^\infty_{\text{loc}}(-\infty, \infty)$.) Then we can write the differential-delay system in the format

$$\dot{x}(t) = (A_0(t) + A_1(t)\sigma_1 + \cdots + A_m(t)\sigma_m)x(t) + B(t)$$

or

$$\dot{x}(t) = A(t, \sigma)x(t) + B(t).$$

Here the entries of the $n \times n$ matrix $A(t, \sigma)$ are polynomials in $\sigma_1, \sigma_2, \ldots, \sigma_m$ (perhaps of high degree in just σ_1 if all delays are integral multiples of h_1), with time-dependent coefficients.

This construction introduces the methods of the theory of polynomial rings and algebraic geometry into the study of differential-delay equations [129]. Such algebraic methods have proved especially useful in the theory of controllability for linear differential-delay systems.

In the general theory of nonlinear differential-hereditary systems for $x \in \mathbf{R}^n$, we encounter equations like $\dot{x}(t) = f(x, x_t)$ (say, autonomous with a finite time-horizon $h > 0$), where $f\colon \mathbf{R}^n \times C \longrightarrow \mathbf{R}^n$ is suitably smooth. Here C is the state space, usually the Banach space of all continuous functions (or perhaps the Sobolev Hilbert space H^1) from the delay interval $-h \leqslant s \leqslant 0$ to $\mathbf{R}^n$. Thus $x_t(s) = x(t + s)$ for $-h \leqslant s \leqslant 0$ defines the state at time $t \geqslant 0$. In this case the dynamical system defines a (local) semi-flow in C as t increases in the future. Since only a future semi-dynamical system is specified, and that holds in an infinite dimensional Banach space, new methods and concepts of topological and differentiable dynamics are demanded [33].

In spite of the many novel difficulties arising in the qualitative study of differential-hereditary systems, there has been developed a valid theory of stability, even hyperbolicity, of critical points and periodic orbits [33]. Also the basic properties of attracting and repelling manifolds have been established and the generic nature of Kupka-Smale structure has been proven [60].

Besides the exploration of the general theory of differential-hereditary systems, a number of serious investigations have illuminated the qualitative behavior of certain explicit nonlinear systems of special interest. For instance, the scalar nonlinear equation [45]

$$\dot{x}(t) = x(t)[a - bx(t - 1)]$$

(and certain closely related generalizations [146]) has a periodic orbit—for appropriate parameters a, b. This equation originally was invented to describe a population growth, where the net-birthrate $[a - bx(t - 1)]$ displays a dependence on the preceding generation. The capacity to utilize information about preceding generations has made the theory of differential-hereditary systems of prime importance in the mathematical theory of population dynamics and ecological interaction of biological species, and the analysis of the spread of epidemics [76, S47].

Queries

In closing this rather lengthy appendix, we list a few open problems that are of great interest within the theory of differentiable dynamics, and also can serve to test the power of new methods.

1) Can the chaos of the generation of periodic orbits of the forced Duffing oscillator be organized and explained by any theory of catastrophes or stochastic dynamics?

2) What are the topological properties of minimal manifolds? In particular can there exist a smooth minimal flow on the sphere S^3?

STOP PRESS: A preprint "There is no minimal flow on S^3", from I. Ishii has just been received containing a proof of the following theorem:

MAIN THEOREM. *If M^3 is an orientable closed 3-dimensional manifold which admits a C^1 minimal flow on it, then its first cohomology group $H^1(M^3; R)$ is not trivial.*

3) Contrary to the preceding question, does every smooth noncritical flow on S^3 have a periodic orbit?

4) Is the C^∞-closing lemma valid for smooth flows (using perturbations and approximations in the appropriate C^∞-topology)?

5) Does structural stability imply Axioms A and B for smooth flows?

6) Can white noise illuminate black holes?

7) When is an elliptic periodic orbit of a Hamiltonian dynamical system Liapunov stable—assuming appropriate generic hypotheses? Is the "drift" of trajectories away from the elliptic periodic orbit a generic phenomenon, regardless of the generic k-jet of the Poincaré map?

Finally, one last query:

What is the difference between the Hopf bifurcation and the Hopf foliation? (reply: EH!).

REFERENCES

Supplemented by bibliographies listed in SURVEYS: Chillingworth 11, Markus 21, 22, Smale 39, and especially Shiraiwa 32.

CONFERENCES: Proceedings and Reports

LECTURE NOTES IN MATHEMATICS, SPRINGER-VERLAG.

C1. Seminar on Differential Equations and Dynamical Systems I and II, Univ. Maryland, 1968 (#60) and 1970 (#144).

C2. Proceedings of the Symposium on Differential Equations and Dynamical Systems, Univ. Warwick, 1969 (#206).

C3. Global Differentiable Dynamics, Case Western Reserve Univ., 1969 (#235).

C4. Japan-United States Seminar on Ordinary Differential and Functional Equations, Kyoto, 1971 (#243).

C5. Conference on the Theory of Ordinary and Partial Differential Equations, Dundee, 1972 (#280).

C6. Stability of Stochastic Dynamical Systems, Univ. Warwick, 1972 (#294).

C7. Symposium on Ordinary Differential Equations, Univ. Minnesota, 1973 (#312).

C8. Recent Advances in Topological Dynamics, Yale Univ., 1972 (#318).

C9. Nonlinear Problems in the Physical Sciences and Biology, Battelle Inst., 1972 (#322).

C10. Dynamical Systems, Univ. Warwick, 1974 (#468).

C11. Structural Stability, the Theory of Catastrophes, and Applications in the Sciences, Battelle Center, 1975 (#525).

C12. Turbulence and Navier-Stokes Equations, Univ. Paris-Sud, 1975 (#565).

C13. Turbulence Seminar, Univ. California, Berkeley, 1977 (#615).

C14. The Structure of Attractors in Dynamical Systems, North Dakota State Univ., 1977 (#668).

C15. Mathematical Control Theory, Canberra, Australia, 1977 (#680).

C16. Equadiff IV, Proceedings, Prague, 1977 (#703).

C17. Ergodic Theory, Proceedings, Oberwolfach (#729).

C18. Functional Differential Equations and Approximations of fixed-points, Proceedings, 1979 (#730).

C19. Global Analysis, Proceedings, 1978 (#755).

C20. International Conference on Dynamical Systems, Northwestern Univ., 1980.

OTHER CONFERENCE PROCEEDINGS OR REPORTS

C21. International Symposium on Nonlinear Oscillations, Kiev, 1969; Poznan, 1973; Berlin, 1977.

C22. Symposium in Pure Mathematics, Global Analysis, vol. 14, Amer. Math. Soc., Providence, R. I., 1970.

C23. NRL Symposium in Differential Equations and Control Theory, Washington, D. C., 1971.

C24. Equadiff III, Purkyne Univ., Brno, 1972.

C25. Global Analysis, ICTP, Trieste, 1972.

C26. Differential-Delay and Functional Equations: Control and Stability, Univ. Warwick, CTC Report #12, 1972.

C27. Symposium on Dynamic Modelling and Control of National Economics, Univ. Warwick, CTC; Inst. Electrical Engineering Conf. Publ. 101, 1973.

C28. International Symposium on Dynamical Systems (Salvador, 1971), Academic Press, 1973.

C29. Symposium on Geometric Methods in System Theory, Proc. NATO Adv. Study Inst., Imperial College, London, Reidel Press, 1973.

C30. Theory and Applications of Differential Games, NATO Adv. Study Inst., Univ. Warwick, CTC; Reidel Press, 1974.

C31. Control Theory and Functional Analysis, ICTP, Trieste, 1974.

C32. Dynamical Systems. Theory and Applications, Lecture Notes in Phys., vol. 38, Springer, 1975.

C33. Manifolds–Tokyo 1973, Univ. Tokyo Press, 1975.

C34. Dynamical Systems. I, II, Academic Press, 1976.

C35. Differential Equations and Topological Dynamics, Univ. Kyoto, 1976.

C36. Dynamical Systems, Univ. Florida, Academic Press, 1977.

C37. Dynamical Systems. I, II, III, Warsaw, Astérisque 49, 50, 51, 1977–1978.

C38. Bifurcation and Catastrophe Theory, New York Acad. Sci., 1977.

C39. Statistical Mechanics and Dynamical Systems, Duke Turbulence Conference, Duke Univ., 1977.

C40. Topics in Nonlinear Dynamics, Tribute to Sir Edward Bullard, Amer. Inst. Phys., 1978.

C41. Applied Nonlinear Analysis, Proc. Conf. Univ. Texas, Academic Press, 1978.

C42. Chaos and Turbulence, New York Acad. Sci., 1979.

C43. Chaos and Strange Attractors, Asilomar Conference, Monterey, California, 1979.

C44. Dynamical Systems, Univ. Nagoya, 1980.

SURVEYS: Books, Monographs, Special Reviews

S1. R. Abraham and J. Marsden, *Foundations of mechanics,* 2nd ed., Benjamin, New York, 1978.

S2. D. V. Anosov and Ja. G. Sinai, *Some smooth ergodic systems,* Russian Math. Surveys 22 (1967), 103–167.

S3. V. I. Arnold, *Small denominators and problems of stability of motion in classical and celestial mechanics,* Russian Math. Surveys 18 (1963), No. 6, 85–192.

S4. ———, *Singularities of smooth mappings,* Russian Math. Surveys 23 (1968), No. 1, 1–43.

S5. ———, *Lectures on bifurcations in versal families,* Russian Math. Surveys 27 (1972), No. 5, 54–124.

S6. ———, *Critical points of smooth functions and their normal forms,* Russian Math. Surveys 30 (1975), No. 5, 1–75.

S7. ———, *Advanced topics in differential equations,* Moscow, 1977.

S8. R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms,* Lecture Notes in Math., vol. 470, Springer-Verlag, Berlin and New York, 1975.

S9. ———, *On Axiom* A *diffeomorphisms,* CBMS Regional Conf. Ser. in Math., no. 35, Amer. Math. Soc., Providence, R. I., 1977.

S10. R. Brockett, *Lie algebras and Lie groups in control theory,* see Conference 29.

S11. D. R. J. Chillingworth, *Differential topology with a view to applications,* London, Pitman, 1976.

S12. J. M. C. Clark, *An introduction to stochastic differential equations on manifolds,* see Conference 29.

S13. G. Debreu, *Four aspects of the mathematical theory of economic equilibrium,* Proc. 1974 Internat. Congr. Math., Vancouver, pp. 65–77.

S14. J. Eells and K. D. Elworthy, *Stochastic differential equations on manifolds* (in preparation).

S15. J. Hale, *Theory of functional differential equations,* Springer, 1977.

S16. A. B. Katok, *Dynamical systems with hyperbolic structures,* Tenth Summer Math. School, Inst. Mat. Ukrain. SSR, Kiev, 1972, pp. 125–211.

S17. A. B. Katok, Ja. G. Sinai and A. M. Stepin, *Theory of dynamical systems and general transformation groups with invariant measure,* J. Soviet Math. 7 (1977), 974–1065.

S18. A. B. Katok and A. M. Stepin, *Approximations in ergodic theory,* Russian Math. Surveys 22 (1967), 77–102.

S19. C. Lobry, *Dynamic polysystems and control theory,* see Conference 29.

S20. E. B. Lee and L. Markus, *Foundations of optimal control theory,* Wiley, New York, 1967.

S21. L. Markus, *Lectures in differentiable dynamics,* CBMS Regional Conf. Ser. in Math., no. 3, Amer. Math. Soc., Providence, R. I., 1971.

S22. ———, *Dynamical systems: Five years after,* see Conference 10.

S23. R. May et al., *Theoretical ecology: Principles and applications,* Blackwell, Oxford, 1976. (See extensive bibliography.)

S24. J. Moser, *Lectures on Hamiltonian systems,* Mem. Amer. Math. Soc. No. 81 (1968).

S25. ———, *Stable and random motions in dynamical systems,* Ann. Math. Studies, no. 77, Princeton Univ. Press, Princeton, N. J., 1973.

S26. J. Moser and C. L. Siegel, *Lectures on celestial mechanics,* Springer, 1971.

S27. Z. Nitecki, *Differentiable dynamics,* MIT Press, Cambridge, Mass., 1971.

S28. M. Peixoto, *On the classification of flows on 2-manifolds,* see Conference 28, pp. 389–419.

S29. Ja. B. Pesin, *Characteristic Lyapunov exponents and smooth ergodic theory,* Russian Math. Surveys 32 (1977), 55–114.

S30. D. Ruelle, *Statistical mechanics and dynamical systems,* see Conference 39.

S31. K. Shell and D. Cass, *Introduction to Hamiltonian dynamics in economics,* The Hamiltonian Approach to Dynamic Economics, Academic Press, New York, 1976.

S32. K. Shiraiwa, *Bibliography of differentiable dynamical systems,* see Conference 44 (extraordinarily complete and extensive bibliography).

S33. M. Shub, *Dynamical systems, filtrations and entropy,* Bull. Amer. Math. Soc. 80 (1974), 27–41.

S34. ———, *The Lefschetz fixed-point formula: Smoothness and stability,* Dynamical Systems, an Internat. Sympos. (Cesari, Hale, La Salle, Eds.), Academic Press, New York, 1976, pp. 13–28.

S35. ———, *Stabilité globale des systèmes dynamiques,* Astérisque 56 (1978).

S36. Ja. Sinai, *Dynamical systems with elastic reflections,* Russian Math. Surveys 25 (1970), 137–189.

S37. ———, *Gibbs measures in ergodic theory,* Russian Math. Surveys 166 (1972), 21–69.

S38. S. Smale, *Differentiable dynamical systems,* Bull. Amer. Math. Soc. 73 (1967), 747–817. (Update version is to appear in following treatise.)

S39. ———, *Mathematics of time* (to appear) (with extensive bibliography).

S40. R. Thom, *Stabilité structurelle et morphogénèse,* Benjamin, New York, 1972.

S41. C. Zeeman, *Catastrophe theory: Selected papers,* Addison-Wesley, Reading, MA, 1977.

CBMS REGIONAL CONF. SER. IN MATH., AMER. MATH. SOC., PROVIDENCE, R. I.

S42. L. Markus, *Lectures in differentiable dynamics,* #3.

S43. H. Blaine Lawson, Jr., *The quantitative theory of foliations,* #27.

S44. A. Weinstein, *Lectures on symplectic manifolds,* #29.

S45. R. Bowen, *On Axiom* A *diffeomorphisms,* #35.

S46. C. C. Conley, *Isolated invariant sets and Morse index,* #38.

S47. F. Hoppensteadt, *Mathematical theories of populations: demographics, genetics and epidemics,* Applied Mathematics, #20.

MEMOIRS AMER. MATH. SOC.

S48. W. Loud, *Periodic solutions of* $x'' + cx' + g(x) = \epsilon f(t)$, #31 (1959).

S49. ———, *Periodic solutions of perturbed second-order autonomous equations,* #47 (1964).

S50. J. Hale and K. Meyer, *A class of functional equations of neutral type,* #76 (1967).

S51. J. Moser, *Lectures on Hamiltonian systems,* #81 (1968).

S52. R. K. Miller and G. Sell, *Volterra integral equations and topological dynamics,* #102 (1970).

S53. L. Markus and K. Meyer, *Generic Hamiltonian dynamical systems are neither integrable nor ergodic,* #144 (1974).

S54. Y. Sibuya, *Uniform simplification in a full neighborhood of a transition point,* #149 (1974).

S55. S. Friedland, *Nonoscillation, disconjugacy, and integral inequalities,* #176 (1976).

S56. R. J. Sacker and G. R. Sell, *Lifting properties in skew-product flows with applications to differential equations,* #190 (1977).

S57. R. Nussbaum, *Differential-delay equations with two time lags,* #205 (1978).

ARTICLES: Research Publications

1. R. Abraham and J. Marsden, *Foundations of mechanics,* 2nd ed., Benjamin, New York, 1978.

2. V. M. Alekseev, *Sur l'allure finale du movement dans le problème des trois corps,* Actes du Congr. Internat. des Math., Vol. 2 (1970), 893–907.

3. D. V. Anosov, *Ergodic properties of geodesic flows on closed Riemannian manifolds of negative curvature,* Soviet Math. Dokl. 4 (1963), 1153–1156.

4. ———, *Geodesic flows on closed Riemannian manifolds with negative curvature,* Proc. Steklov Inst. Math. 90 (1967), 1–235.

5. ———, *Existence of smooth ergodic flows on smooth manifolds,* Math. USSR Izv. 8 (1974), 525–552.

6. D. V. Anosov and A. B. Katok, *New examples in smooth ergodic theory, ergodic diffeomorphisms,* Trans. Moscow Math. Soc. 23 (1970), 1–35.

7. R. Armstrong and R. McGehee, *Some mathematical problems concerning the ecological principle of competitive exclusion,* J. Differential Equations 23 (1976), 30–52.

8. V. I. Arnold, *Proof of a theorem of A. N. Kolmogorov on the invariance of quasi-periodic motions under small perturbations of the Hamiltonian,* Russian Math. Surveys 18 (1963), No. 5, 9–36.

9. ———, *Small denominators and problems of stability of motion in classical and celestial mechanics,* Russian Math. Surveys 18 (1963), No. 6, 85–192.

10. D. Asimov, *Homotopy of non-singular vector fields to structurally stable ones,* Ann. of Math. (2) 102 (1975), 55–65.

11. R. Bowen, *Entropy versus homology for certain diffeomorphisms,* Topology 13 (1974), 61–67.

12. ———, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms,* Lecture Notes in Math., vol. 470, Springer-Verlag, Berlin and New York, 1975.

13. ———, *On Axiom* A *diffeomorphisms,* CBMS Regional Conf. Ser. in Math., no. 35, Amer. Math. Soc., Providence, R. I., 1977.

14. R. Bowen et al., Publication of Inst. Hautes Études Sci. Publ. Math., Dedicated to R. Bowen, vol. 50, 1979.

15. R. Bowen and D. Ruelle, *The ergodic theory of Axiom* A *flows,* Invent. Math. 29 (1975), 181–202.

16. R. Brockett, *Lie algebras and Lie groups in control theory,* see Conference 29.

17. J. L. Callot, F. Diener and M. Diener, *Le problème de la Chasse au canard,* C. R. Acad. Sci. Paris Ser. A 286 (1978), 1059–1061.

18. C. Camacho, *Morse-Smale* $\mathbf{R}^2$*-actions on two-manifolds,* see Conference 28, pp. 71–74.

19. S. N. Chow, J. K. Hale and J. Mallet-Paret, *Applications of generic bifurcation.* I, Arch. Rat. Mech. Anal. 59 (1975), 159–188.

20. J. M. C. Clark, *An introduction to stochastic differential equations on manifolds,* see Conference 29.

21. F. Diener, *Les équations* $\epsilon\ddot{x} + (x^2 - 1)\dot{x}^{[s]} + x = a$, Collect. Math. (1979).

22. R. Driver, *Ordinary and delay differential equations,* Springer, 1977.

23. K. D. Elworthy, *Stochastic dynamical systems and their flows,* Conference on Stochastic Analysis, Academic Press, 1978, pp. 79–95.

24. D. B. A. Epstein, *Periodic flows on three-manifolds,* Ann. of Math. (2) 95 (1972), 66–82.

25. A. Fathi and M. Herman, *Existence de diffeomorphismes minimaux,* Astérisque 49 (1977), 37–59.

26. J. Franks, *The structure of Smale diffeomorphisms,* see Conference 14, pp. 117–126.

27. J. Franks and R. F. Williams, *Anomalous Anosov flows* (preprint, 1979).

28. M. Golubitsky, *An introduction to catastrophe theory and its applications,* SIAM Rev. (1977) (contains extensive bibliography).

29. J. Guckenheimer, *Bifurcation and catastrophe,* see Conference 28, pp. 95–109.

30. ———, *On the bifurcation of maps of the interval,* Invent. Math. 39 (1977), 165–178.

31. ———, *Structural stability of the Lorenz attractors* (preprint, 1979).

32. ———, *Sensitive dependence upon initial conditions for one-dimensional maps* (preprint, Inst. Hautes Études Sci. Publ. Math., 1979).

33. J. K. Hale, *Theory of functional differential equations,* 2nd ed., Springer, 1977.

34. J. Harrison, *Unsmoothable diffeomorphisms on higher dimensional manifolds,* Proc. Amer. Math. Soc. 73 (1979), 249–255.

35. M. Herman, *Sur la conjugaison différentiable des diffeomorphismes du cercle à des rotations,* Inst. Hautes Études Sci. Publ. Math. 49 (1979), 5–234.

36. M. Hirsch and C. Pugh, *Stable manifolds and hyperbolic sets,* Global analysis, Conference 22, pp. 133–163.

37. M. Hirsch, C. Pugh and M. Shub, *Invariant manifolds,* Lecture Notes in Math., vol. 583, Springer-Verlag, Berlin and New York, 1977.

38. E. Hopf, *Abzweigung einer Periodischen Lösung eines Differential Systems,* Ber. Verh. Sächs. Akad. Wiss. Leibzig Math. Phys. 95 (1943), 3–22.

39. H. Hopf, *Über die Abbildungen der 3-sphären auf die Kugelflache,* Math. Ann. 104 (1931), 637–665.

40. I. Ishii, *On a non-homogeneous flow on the 3-dimensional torus,* Funct. Equations 17 (1974), 231–248. (See also Conference 35.)

41. ———, *There is no minimal flow on* S^3 (preprint, 1980).

42. L. Jonker and D. Rand, *Bifurcations in one dimension* I: *The nonwandering set* (preprint, 1980).

43. V. Jurdjevic and H. J. Sussmann, *Control systems on Lie groups,* J. Differential Equations 12 (1972), 313–329.

44. ———, *Controllability of nonlinear systems,* J. Differential Equations 12 (1972), 95–116.

45. S. Kakutani and L. Markus, *On a nonlinear difference-differential equation,* Contributions to the Theory of Nonlinear Oscillations, Vol. IV, Ann. of Math. Studies, no. 41, Princeton Univ. Press, Princeton, N. J. , 1958, pp. 1–18. (See also Conference 26.)

46. R. E. Kalman, *Lectures on controllability and observability,* CIME Summer Course, 1968, Cremonese, Roma, 1969.

47. ———, *On the realization of multilinear response maps,* Ricerche di Automatica (to appear).

48. A. B. Katok, *Hyperbolicity, entropy and minimality for smooth dynamical systems,* Univ. Maryland, 1979, preprint.

49. ———, *Lyapunov exponents, entropy and periodic orbits of diffeomorphisms,* Univ. Maryland, 1979, preprint.

50. A. N. Kolmogorov, *A general theory of dynamic systems and classical mechanics,* Proc. Internat. Congr. Math. I (1954), 315–333.

51. M. Levi, *Qualitative analysis of the periodicity forced relaxation oscillations,* Thesis N. Y. U., 1978. "Relaxation oscillation with periodic forcing" (see Conference 20).

52. T. Y. Li and J. A. Yorke, *Period three implies chaos,* Amer. Math. Monthly 82 (1975), 985.

53. C. Lobry, *Dynamic polysystems and control theory,* see Conference 29.

54. E. N. Lorenz, *Deterministic nonperiodic flow,* J. Atmospheric Sci. 20 (1963), 130–141.

55. W. Loud, *Periodic solutions of perturbed second-order autonomous equations,* Mem. Amer. Math. Soc. No. 47 (1964).

56. R. McGehee, *Parabolic orbits in the three-body problem,* see Conference 28.

57. ———, *Singularities in classical celestial mechanics,* Proc. 1978 Internat. Congr. Math., Helsinki.

58. R. McGehee and J. N. Mather, *Solutions of the colinear four-body problem which become unbounded in finite time,* see Conference 32, pp. 573–597.

59. R. McGehee and K. Meyer, *Homoclinic points of area-preserving diffeomorphisms,* Amer. J. Math. 96 (1974), 409–421.

60. J. Mallet-Paret, *Generic properties of differential functional equations,* Thesis, Univ. Minnesota, 1974.

61. J. Mallet-Paret and J. Yorke, *Snakes: oriented families of periodic orbits, their sources, sinks, and continuation* (preprint, 1979). (See also Conference 42.)

62. R. Mañé, *Contributions to the stability conjecture,* Topology 17 (1978), 383–396.

63. A. Manitius and R. Triggiani, *Function space controllability of linear retarded systems: a derivation from abstract operator conditions,* SIAM J. Control and Optimization 16 (1978), 599–645.

64. A. Manning, *Axiom* A *diffeomorphisms have rational zeta functions,* Bull. London Math. Soc. 3 (1971), 215–220.

65. ———, *Anosov diffeomorphisms on nilmanifolds,* Proc. Amer. Math. Soc. 38 (1973), 423–426.

66. ———, *There are no new Anosov diffeomorphisms on tori,* Amer. J. Math. 96 (1974), 422–429.

67. ———, *Topological entropy and the first homology group,* see Conference 10, pp. 185–190.

68. ———, *Topological entropy and homology,* Astérisque 40 (1976), 141–142.

69. L. Markus, *Lie dynamical systems,* Amer. Math. Soc. address, 1972 (to appear within next decade).

70. ———, *Controllability in topological dynamics,* Proc. 1974 Internat. Congr. Math., Vancouver, pp. 355–360.

71. ———, *Notes on geometrical problems in global stochastic dynamics,* Research report, Imperial College, 1978, pp. 1–26.

72. L. Markus and K. Meyer, *Generic Hamiltonian dynamical systems are neither integrable nor ergodic,* Mem. Amer. Math. Soc. No. 144 (1974).

73. ———, *Periodic orbits and solenoids in generic Hamiltonian dynamical systems,* Amer. J. Math. 102 (1980), 25–92.

74. L. Markus and G. Sell, *Control in conservative dynamical systems: Recurrence and capture in aperiodic fields,* J. Differential Equations 16 (1974), 472–505.

75. F. R. Marotto, *Snap-back repellers imply chaos in* $\mathbf{R}^n$, J. Math. Anal. Appl. 63 (1978), 199–223.

76. R. M. May, *Simple mathematical models with very complicated dynamics,* Nature 261 (1976), 459–467.

77. K. Meyer, *On the convergence of the zeta functions for flows and diffeomorphisms,* J. Differential Equations 5 (1969), 338–345.

78. J. Milnor and W. Thurston, *On iterated maps of the interval.* I, II, Princeton Univ., 1977 (preprint).

79. J. W. Morgan, *Non-singular Morse-Smale flows on 3-dimensional manifolds,* Topology 18 (1979), 41–53.

80. G. Morris, *Bounds on solutions of forced conservative oscillators,* see Conference 23; also Proc. 1970 Internat. Congr., Nice.

81. A. S. Morse, *Ring models for delay-differential systems,* Proc. IFAC Sympos. on Multivariable Technological Systems, Manchester, United Kingdom, pp. 561–567; also Automatica 12 (1976), 529–531.

82. J. Moser, *On invariant curves of area preserving mappings of an annulus,* Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl II (1962), 1–20.

83. ———, *Lectures on Hamiltonian systems,* Mem. Amer. Math. Soc. No. 81 (1968).

84. ———, *Stable and random motions in dynamical systems,* Ann. Math. Studies, no. 77, Princeton Univ. Press, Princeton, N. J., 1973.

85. S. Newhouse and J. Palis, *Bifurcations of Morse-Smale dynamical systems,* see Conference 28, pp. 303–366.

86. ———, *Cycles and bifurcation theory,* Astérisque 31 (1976), 43–140.

87. S. Newhouse and M. Peixoto, *There is a simple arc joining any two Morse-Smale flows,* Astérisque 31 (1976), 15–41.

88. J. Palis, *Arcs of dynamical systems: bifurcation and stability,* see Conference 10, pp. 48–52.

89. ———, *On Morse-Smale dynamical systems,* Topology 8 (1969), 385–405.

90. W. Parry, *Entropy and generators in ergodic theory,* Benjamin, New York, 1969.

91. ———, *The Lorenz attractor and a related population model,* see Conference 17, pp. 169–187.

92. M. M. Peixoto, personal communication.

93. C. C. Pugh, *On the entropy conjecture,* see Conference 10, pp. 257–261.

94. C. Pugh and C. Robinson, *The* C^1 *closing lemma, including Hamiltonians.*

95. D. Rand, *The topological classification of Lorenz attractors,* Proc. Cambridge Philos. Soc. 83 (1978), 451–460.

96. B. Raymond, *Poincaré-Bendixson theorem does not hold for foliated 3-manifolds,* Report at Oberwolfach Conference, 1971.

97. J. Robbin, *A structural stability theorem,* Ann. of Math. (2) 94 (1971), 447–493.

98. R. C. Robinson, *Generic properties of conservative systems.* I, II, Amer. J. Math. 92 (1970), 562–603; 897–906.

99. ———, *A global approximation theorem for Hamiltonian systems,* Global analysis, see Conference 22, pp. 233–243.

100. R. C. Robinson, *Structural stability of vector fields,* Ann. of Math. (2) 99 (1974), 154–175.

101. ———, *Structural stability of C^1 flows,* see Conference 10, pp. 262–277.

102. ———, *Structural stability of C^1 diffeomorphisms,* J. Differential Equations 22 (1976), 28–73.

103. D. Ruelle, *A measure associated with Axiom* A *attractors,* Amer. J. Math. 98 (1976), 619–654.

104. ———, *Statistical mechanics and dynamical systems,* see Conference 39, pp. 1–31.

105. D. Saari, *Singularities of Newtonian gravitational systems,* see Conference 28, pp. 479–488.

106. B. Schmitt, *Détermination, à l'aide d'index, des solutions periodiques de l'équation de Duffing* $x'' + 2x^3 = \lambda \cos t$, see Conference 5, pp. 330–334.

107. P. A. Schweitzer, *Counterexamples to the Seifert conjecture and opening closed leaves of foliations,* Ann. of Math. (2) 100 (1974), 386–400.

108. G. Sell, *The structure of a flow in the vicinity of an almost periodic motion,* J. Differential Equations 27 (1978), 359–393.

109. ———, *Bifurcation of higher dimensional tori,* Arch Rat. Mech. (to appear).

110. K. Shell and D. Cass, *Introduction to Hamiltonian dynamics in economics,* The Hamiltonian Approach to Dynamic Economics, Academic Press, New York, 1976.

111. K. Shiraiwa, *Some conditions on Anosov diffeomorphisms,* see Conference 33, pp. 205–209.

112. ———, *Boundedness and convergence of solutions of Duffing's equation,* Nagoya Math. J. 66 (1977), 151–166. (See also Conference 35.)

113. K. Shiraiwa and M. Kurata, *A generalization of a theorem of Marotto,* Proc. Japan Acad. Ser. A 55 (1979), 286–289.

114. M. Shub, *Topologically transitive diffeomorphisms of* T^4, see Conference 2, pp. 39–40.

115. Y. Sibuya, *Global theory of a second order linear ordinary differential equation with a polynomial coefficient,* North-Holland Math. Studies, no. 18, North-Holland, Amsterdam, 1975.

116. Y. Sibuya and J. Kato, *Catastrophic deformation of a flow and non-existence of almost periodic solutions,* J. Fac. Sci. Univ. Tokyo 24 (1977), 267–280.

117. Ya. G. Sinai, *Geodesic flows on compact surfaces of negative curvature,* Soviet Math. Dokl. 2 (1961), 106–109.

118. ———, *On the foundations of the ergodic hypothesis for a dynamical system of statistical mechanics,* Soviet Math. Dokl. 4 (1963), 1818–1822.

119. ———, *Ergodic properties of a gas in one-dimensional hard globules with an infinite number of degrees of freedom,* Functional Anal. Appl. 6 (1972), 41–50.

120. ———, *Gibbs measures in ergodic theory,* Russian Math. Surveys 166 (1972), 21–69.

121. S. Smale, *Diffeomorphisms with many periodic points,* Differential and combinatorial topology, Princeton Univ. Press, Princeton, N. J., 1964.

122. S. Smale, *Optimizing several functions,* see Conference 33 (1973), pp. 69–75.

123. ———, *Global analysis and economics.* I, *Pareto optimum and a generalization of Morse theory,* see Conference 28 (1973).

124. ———, *Global analysis and economics.* IIA, *Extension of a theorem of Debreu,* J. Math. Econom. 1 (1974), 1–14.

125. ———, *Global analysis and economics.* III, *Pareto optima and price equilibria,* J. Math. Econom. 1 (1974), 107–117.

126. ———, *Global analysis and economics.* IV, *Finiteness and stability of equilibria with general consumption sets and production,* J. Math. Econom. 1 (1974), 119–127.

127. ———, *Global analysis and economics.* V, *Pareto theory with constraints,* J. Math. Econom. 1 (1974), 213–221.

128. ———, *Global analysis and economics.* VI, *Geometric analysis of Pareto optima and price equilibria under classical hypotheses,* J. Math. Econom. 3 (1976), 1–14.

129. E. D. Sontag, *Linear systems over commutative rings: A survey,* Ricerche di Automatica 7 (1976), 1–34.

130. J. Sotomayor, *Generic bifurcations of dynamical systems,* see Conference 28 (1971), pp. 561–582.

131. D. Sullivan, *A counterexample to the periodic orbit conjecture,* Inst. Hautes Études Sci. Publ. Math. 46 (1976), 5–14.

132. D. Sullivan and R. F. Williams, *On the homology of attractors,* Topology 15 (1976), 259–262.

133. F. Takens, *Unfoldings of certain singularities of vector fields: Generalized Hopf bifurcations,* J. Differential Equations 14 (1973), 476–493.

134. ———, *Normal forms for certain singularities of vector fields,* Ann. Inst. Fourier (Grenoble) 23 (1973), 163–195.

135. R. Thom, *Stabilité structurelle et morphogénèse,* Benjamin, New York, 1972.

136. W. Thurston, *Foliations of codimension one* (preprint).

137. ———, *Foliations and group of diffeomorphisms,* Bull. Amer. Math. Soc. 80 (1974), 304–307.

138. Y. Ueda, *Randomly transitional phenomena in the system governed by Duffing's equation,* J. Statist. Phys. 20 (1979), 181–196. (See also Conferences 42 and 43.)

139. J. Vey, *Sur les orbites périodique des systèmes dynamique au voisinage d'un point d'équilibre,* C. R. Acad. Sci. Paris Ser. A-B 283 (1976).

140. P. Walters, *Ergodic theory-introductory lectures,* Lecture Notes in Math., vol. 458, Springer-Verlag, Berlin and New York, 1975.

141. A. Weinstein, *Lagrangian submanifolds and Hamiltonian systems,* Ann. of Math. (2) 98 (1973), 377–410.

142. ———, *Normal models for nonlinear Hamiltonian systems,* Invent. Math. 20 (1973), 47–57.

143. R. F. Williams, *Cohomology of expanding attractors,* see Conference 10, p. 41.

144. ———, *The structure of Lorenz attractors,* Lecture Notes in Math., vol. 615, 1977; see Conference 13, pp. 94–115.

145. R. F. Williams, *The structure of Lorenz attractors,* Northwestern Univ., 1976 (preprint); Inst. Hautes Études Sci. Publ. Math. (to appear).

146. J. Yorke, *Stability of a periodic solution,* see Conference 26 (1972).

147. E. C. Zeeman, C^0*-density of stable diffeomorphisms and flows,* Southhampton Colloq. (1972).

148. ———, *Catastrophe theory: Selected papers,* Addison-Wesley, Reading, Mass., 1977.

149. E. C. Zeeman and D. Trotman, *The classification of elementary catastrophes of codimension* $\leqslant 5$, Warwick Math. Report, 1974; also see Lecture Notes in Math., vol. 525, Springer-Verlag, Berlin and New York, 1976.

ABCDEFGHIJ–AMS–89876543210

ISBN 978-0-8218-1695-0

9 780821 816950

CBMS/3