Conference Board of the Mathematical Sciences
REGIONAL CONFERENCE SERIES IN MATHEMATICS

Number 4

H. S. M. Coxeter

Twisted honeycombs

Supported by the National Science Foundation
Published by the American Mathematical Society

http://dx.doi.org/10.1090/cbms/004

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by

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Providence, Rhode Island

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International Standard Book Number 0-8218-1653-5
Library of Congress Catalog Card Number 75-145638
AMS 1970 Subject Classifications: 05B30, 05B45, 05C25, 16A40, 20B25, 20F05, 20H15, 50A10, 50B30, 50C15, 55A15, 57A05, 57A10.

Key Words and Phrases: dodecahedron spaces, finite groups of quaternions, finite groups of rotations, fundamental group of a manifold, fundamental region, glide, hyperbolic geometry, linear fractional group, octahedron space, orthoscheme, Petrie polygon, Platonic solids and their symmetry groups, presentation of a group, pure quaternion, reflection groups, regular honeycomb, regular polytope, right and left screws, Schläfli symbol, simplex, spherical geometry, torus, truncated cube space, vertex figure.

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PREFACE

Following D. M. Y. Sommerville [7, p. 166] we define a *honeycomb* to be a symmetrical subdivision of a three-dimensional manifold into a number of polyhedral cells, all alike, each rotation that is a symmetry operation of a cell being also a symmetry operation of the whole configuration. The honeycomb is said to be *twisted* if it is not symmetrical by reflection; thus a twisted honeycomb, like a screw, occurs in right-handed and left-handed varieties. The subject originated in 1933, when Weber and Seifert [10] considered two one-celled honeycombs, each consisting of a regular dodecahedron with opposite faces identified. The work involves a combination of geometry and group theory, assisted by quaternions.

I wish to thank the National Science Foundation and the University of Maine for sponsoring this series of lectures, and to thank Norman W. Johnson and Grattan P. Murphy for taking notes. I gratefully acknowledge also the help of John Leech and Abraham Sinkov, who computed the orders of certain groups presented in the form

$$L^2 = M^2 = N^2 = (LM)^p = (MN)^r = (LN)^t = (LMN)^q = 1,$$

and found representations for these groups.

H. S. M. Coxeter
February 1970

http://dx.doi.org/10.1090/cbms/004/17

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