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Number 5

George W. Whitehead

Recent advances in homotopy theory

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by

George W. Whitehead

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CONTENTS

PREFACE

At the end of the second World War, the subject of homotopy theory was in a very rudimentary state. Since then its development has been very rapid, and it has had a profound effect on other branches of topology. In fact, the reduction of geometric questions to problems in homotopy theory is almost "standard operating procedure". Examples of this approach are the classification of fibre bundles and the determination of the ring of cobordism classes of manifolds.

Early in the history of the subject it was observed that many phenomena became much more regular in the "stable range". This has led to a particularly rapid development of stable homotopy theory as a subject in its own right. Yet, despite all the great advances that have been made, such is our ignorance that there is not a single stably nontrivial finite complex whose stable homotopy groups are completely known.

These lectures are intended to describe some of these advances, particularly in the stable theory. Because of the breadth of the subject, I have made no attempt at completeness, but have chosen instead to discuss those topics which have been closest to my own interests.

Chapter I is devoted to general homology and cohomology theories, pairings, and duality theorems. In Chapter II, semi-simplicial spectra are introduced as a setting for my joint work with Kan on Poincaré duality. Chapter III discusses the Adams spectral sequence and some of its recent unstable analogues. Chapter IV is devoted to stable homotopy groups as modules over the stable homotopy ring σ_*, and Chapter V to some of the module extensions which arise naturally in the subject.

These lectures were delivered at the 1969–1970 Holiday Mathematics Symposium at the New Mexico State University at Las Cruces, as part of the Regional Conference Program sponsored by the Conference Board of the Mathematical Sciences with the support of the National Science Foundation. I wish to express my appreciation to the Conference Board for making them possible, and to the Department of Mathematical Sciences of the University for their invitation, as well as for their lavish hospitality during the conference.

George W. Whitehead

Massachusetts Institute of Technology
June 16, 1970

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TABLE I. The unstable Adams spectral sequences for spheres.

9														$(121124111)_{23124111}$		9
8												$(11124111)_{2224111}$	$(12124111)_{2324111}$ $(21124111)_{4124111}$	$(31124111)_{4224111}$	$(h_0^7h_4)$ 41124111	8
7											1124111 $(P^1h_0^2h_2)$	$(1224111)_{244111}$ $(2124111)_{424111}$	$(1211233)_{231233}$ $(3124111)_{524111}$	$(3224111)_{444111}$	$(h_0^6h_4)$ 5124111	7
6										124111 $(P^1h_1^2)$	$(111233)_{22233}$ 224111 $(P^1h_0h_2)$	$(121233)_{23233}$ $(211233)_{41233}$ $(324111)_{64111}$	$(311233)_{42233}$	$(h_0^2d_0)$ 344111	$(123333)_{23533}$ $(h_0^5h_4)$ 624111	6
5						$(12111)_{2311}$			24111 (P^1h_1)	$(11233)_{8111}$ $(34111)_{4511}$	$(12233)_{2433}$ $(21233)_{4233}$ 44111 (P^1h_2)	$(12333)_{2353}$ $(31233)_{5233}$ $(36111)_{4711}$ $(54111)_{6511}$	$(32233)_{4433}$	$(23333)_{4533}$ 51233 (h_0d_0) $(74111)_{8511}$	24333 (h_1d_0) $(61233)_{8233}$ 84111 $(h_0^4h_4)$	5
4				$(1111)_{221}$	$(1211)_{231}$ $(2111)_{411}$	$(3111)_{421}$	$(h_0^3h_3)$ 4111	$(5111)_{621}$	1233 (h_1c_0) $(6111)_{811}$	$(2233)_{911}$ $(3511)_{461}$ $(7111)_{821}$	$(2333)_{453}$ $(3233)_{633}$ $(3611)_{471}$	$(3333)_{553}$ $(5511)_{661}$ $(9111)_{\underline{10}21}$	$(3433)_{473}$ $(4333)_{833}$ $(\underline{10}111)_{\underline{12}11}$	$(3533)_{573}$ $(7511)_{861}$ $(5333)_{933}$ $(\underline{11}111)_{\underline{12}21}$ 6233 (d_0)	$(6333)_{853}$ $(7233)_{\underline{10}33}$ $\underline{12}111$ $(h_0^3h_4)$	4
3			111 $(h_0^2h_2)$	$(121)_{23}$ $(211)_{41}$	$(311)_{51}$	$(321)_{43}$	$(h_0^2h_3)$ 511	233 (c_0) $(521)_{63}$ $(611)_{81}$	333 (h_2^3) $(711)_{91}$	$(361)_{47}$ $(433)_{\underline{10}1}$ $(721)_{83}$	$(353)_{57}$ $(533)_{93}$	$(561)_{67}$ $(921)_{\underline{10}3}$ $(\underline{10}11)_{\underline{12}1}$	$(733)_{\underline{11}3}$ $(\underline{11}11)_{\underline{13}1}$	653 $(h_0h_3^2)$ $(761)_{87}$ $(\underline{11}21)_{\underline{12}3}$	$(753)_{97}$ $\underline{13}11$ $(h_0^2h_4)$	3
2	$(10)_2$	(h_1^2) 11	21 (h_0h_2) $(30)_4$	$(31)_5$	$(50)_6$	(h_2^2) 33	61 (h_0h_3) $(70)_8$	53 (h_1h_3) $(71)_9$	$(90)_{\underline{10}}$	$(73)_{\underline{11}}$	$(\underline{11}0)_{\underline{12}}$	$(\underline{11}1)_{\underline{13}}$	$(\underline{13}0)_{\underline{14}}$	77 (h_3^2)	$\underline{14}1$ (h_0h_4) $(\underline{15}0)_{\underline{16}}$	2
1	1 (h_1)		3 (h_2)				7 (h_3)								$\underline{15}$ (h_4)	1
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	

TABLE I, Section 1

13							$(12112411124111)_{2112411124111}$	13
12					$(111241124111)_{2224 1124111}$	$(121241124111)_{2324 1124111}$ $(211241124111)_{4124112411}$	$(311241124111)_{4224 1124111}$	12
11				$11241124111\,(P^2h_0^2h_2)$	$(12241124111)_{2441124111}$ $(21241124111)_{4241124111}$	$(12112344111)_{2312344111}$ $(31241124111)_{5241124111}$	$(32241124111)_{4441124111}$	11
10			$1241124111\quad(P^2h_1^2)$	$(1112344111)_{222344111}$ $2241124111\,(P^2h_0h_2)$	$(1212344111)_{232344111}$ $(2112344111)_{412344111}$ $(3241124111)_{641124111}$	$(3112344111)_{422344111}$	$3441124111\quad(P^1h_0^2d_0)$	10
9		(P^2h_1) 241124111	$(112344111)_{81124111}$ $(341124111)_{45124111}$	$(122344111)_{24344111}$ $(212344111)_{42344111}$ $441124111\,(P^2h_2)$	$(121124333)_{23124333}$ $(312344111)_{52344111}$ $(361124111)_{47124111}$ $(541124111)_{65124111}$	$(322344111)_{44344111}$	$512344111\quad(P^1h_0d_0)$ $(741124111)_{85124111}$	9
8	$(51124111)_{6224111}$	$(P^1h_1c_0)$ 12344111 $(61124111)_{8124111}$	$(11124333)_{2224333}$ $(22344111)_{9124111}$ $(35124111)_{4624111}$ $(71124111)_{8224111}$	$(12124333)_{2324333}$ $(21124333)_{4124333}$ $(32344111)_{6344111}$ $(36124111)_{4724111}$	$(31124333)_{4224333}$ $(55124111)_{6624111}$ $(91124111)_{\underline{10}224111}$	$(34344111)_{4744111}$ $(41124333)_{8344111}$ $(\underline{10}1124111)_{\underline{12}124111}$	$(51124333)_{6224333}$ $62344111\quad(P^1d_0)$ $(75124111)_{8624111}$ $(\underline{11}1124111)_{\underline{12}24111}$	8 $\underline{12}1124111$
7	$2344111\quad(P^1c_0)$ $(5224111)_{644111}$ $(6124111)_{824111}$	$(h_0^3e_0)$ 1124333 $(3344111)_{451233}$ $(7124111)_{924111}$	$(1224333)_{236233}$ $(2124333)_{424333}$ $(4344111)_{\underline{10}24111}$ $(3624111)_{484111}$ $(7224111)_{844111}$	$(3124333)_{524333}$ $(3544111)_{471233}$ $(5344111)_{651233}$	$(3224333)_{436233}$ $(5624111)_{684111}$ $(9224111)_{\underline{10}44111}$ $(\underline{10}124111)_{\underline{12}24111}$	$(1245333)_{247333}$ $(5124333)_{\underline{11}44111}$ $(7344111)_{851233}$ $(\underline{11}124111)_{\underline{13}24111}$	$(2245333)_{\underline{14}24111}$ $(7624111)_{884111}$ $(5224333)_{636233}$ $(\underline{11}224111)_{\underline{12}44111}$ $(6124333)_{824333}$	7 $\underline{13}124111$

TABLE I, Section 2

	16	17	18	19	20	21	22	
6	124333 $(h_1^2 d_0)$ $(544111)_{81233}$ $(724111)_{\underline{10}4111}$	$(h_0^2 e_0)$ 224333 $(351233)_{46233}$	$(324333)_{56233}$ $(361233)_{47233}$ $(744111)_{\underline{12}4111}$	$(551233)_{66233}$	245333 $(h_0^2 g)$ $(944111)_{\underline{12}1233}$ $(336233)_{48333}$ $(\underline{11}24111)_{\underline{14}4111}$ $(584111)_{6\underline{12}111}$	$(345333)_{45733}$ $(624333)_{\underline{13}1233}$ $(751233)_{86233}$	$(445333)_{\underline{16}4111}$ $(536233)_{68333}$ $(724333)_{96233}$ $(784111)_{8\underline{12}111}$	6 545333
5	$(34333)_{4733}$ $(71233)_{9233}$ $(94111)_{\underline{10}511}$	$(35333)_{4653}$ $(h_0 e_0)$ 36233	$(36333)_{4753}$ $(91233)_{\underline{16}111}$ 45333 $(h_0 f_0)$ $(\underline{11}4111)_{\underline{12}511}$ $(5\underline{10}111)_{6\underline{11}11}$	$(\underline{10}1233)_{\underline{12}233}$	47333 $(h_0 g)$ $(\underline{11}1233)_{\underline{13}233}$ $(\underline{13}4111)_{\underline{14}511}$	35733 $(h_1 g)$ $(76233)_{\underline{14}233}$	$(58333)_{\underline{20}111}$ $(7\underline{12}111)_{8\underline{13}11}$ $(1\underline{5}4111)_{\underline{16}511}$	5 55733
4	$(7333)_{953}$ $(9511)_{\underline{10}61}$ $(\underline{13}111)_{\underline{14}21}$	$(3653)_{477}$ $(5\underline{10}11)_{6\underline{11}1}$ 8333 (e_0) $(\underline{14}111)_{\underline{16}11}$	5733 (f_0) $(\underline{11}511)_{\underline{12}61}$ 9333 $(h_0^2 h_2 h_4)$ $(\underline{15}111)_{\underline{16}21}$ $(\underline{10}233)_{\underline{17}11}$	$(5653)_{677}$ $(\underline{10}333)_{\underline{12}53}$ $(\underline{11}233)_{\underline{14}33}$	6653 (g) $(\underline{13}511)_{\underline{14}61}$ $(\underline{11}333)_{\underline{13}53}$ $(1\underline{7}111)_{\underline{18}21}$	$(7653)_{877}$ $(\underline{12}333)_{\underline{16}33}$ $(\underline{18}111)_{\underline{20}11}$	3577 $(h_2 c_1)$ $(1\underline{3}333)_{1\underline{7}33}$ $(7\underline{13}11)_{8\underline{14}1}$ $(\underline{15}511)_{\underline{16}61}$ $(8653)_{\underline{21}11}$ $(\underline{19}111)_{\underline{20}21}$	4
3	$(961)_{1\underline{0}7}$ $(\underline{13}21)_{\underline{14}3}$ $(\underline{14}11)_{\underline{16}1}$	$\underline{11}33$ $(h_1^2 h_4)$ $(\underline{15}11)_{\underline{17}1}$	$\underline{10}53$ $(h_0 h_2 h_4)$ $(\underline{12}33)_{\underline{18}1}$ $(\underline{11}61)_{\underline{12}7}$ $(\underline{15}21)_{\underline{16}3}$	577 (c_1) $(\underline{11}53)_{\underline{13}7}$ $(\underline{13}33)_{\underline{17}3}$	$(\underline{13}61)_{\underline{14}7}$ $(\underline{17}21)_{\underline{18}3}$ $(\underline{18}11)_{\underline{20}1}$	777 (h_3^3) $(\underline{15}33)_{\underline{19}3}$ $(1\underline{9}11)_{\underline{21}1}$	$(7\underline{14}1)_{8\underline{15}}$ $(1\underline{5}61)_{\underline{16}7}$ $(\underline{14}53)_{\underline{22}1}$ $(\underline{19}21)_{\underline{20}3}$	3
2	$\underline{13}3$ $(h_1 h_4)$ $(\underline{15}1)_{\underline{17}}$	$(\underline{17}0)_{\underline{18}}$	$\underline{11}7$ $(h_2 h_4)$ $(\underline{15}3)_{\underline{19}}$	$(\underline{19}0)_{\underline{20}}$	$(\underline{19}1)_{\underline{21}}$	$(\underline{21}0)_{\underline{22}}$	$(\underline{15}7)_{\underline{23}}$	2
1								1
	16	17	18	19	20	21	22	

TABLE I, Section 3

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TABLE II. The Adams spectral sequence for S^0.

	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	
15												$h_0^2P^3h_2$				15
14											$h_1P^3h_1$	$h_0P^3h_2$			$h_0^2P^2d_0$	14
13										P^3h_1		P^3h_2			$h_0P^2d_0$	13
12								h_0^5i		$h_1P^2c_0$					P^2d_0	12
11				$h_0^2P^2h_2$				h_0^4i	P^2c_0	$h_0^3P^1e_0$					h_0^5r	11
10			$h_1P^2h_1$	$h_0P^2h_2$			$h_0^2P^1d_0$	h_0^3i	$h_1^2P^1d_0$	$h_0^2P^1e_0$			$h_0^2P^1g$		h_0^4r	10
9		P^2h_1		P^2h_2			$h_0P^1d_0$	$h_1P^1d_0$ h_0^2i		$h_0P^1e_0$	h_0^2j		h_0P^1g	h_0^2k	h_0^3r	9
8		$h_1P^1c_0$					P^1d_0	h_0i		P^1e_0	h_0j		P^1g	h_0k	h_0^2r	8
7	P^1c_0	$h_0^3e_0$						i			j			k	h_0r	7
6	$h_1^2d_0$	$h_0^2e_0$			h_0^2g			h_0h_2g			h_2^2g				r	6
5		h_0e_0	h_0f_0		h_0g	h_1g		h_2g	$h_1h_4c_0$						$h_0^3h_4^2$	5
4		e_0	f_0 $h_0^2h_2h_4$		g		h_2c_1	h_4c_0							$h_0^2h_4^2$	4
3		$h_1^2h_4$	$h_0h_2h_4$	c_1		h_3^3									$h_0h_4^2$	3
2	h_1h_4		h_2h_4												h_4^2	2
1																1

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
8	h_0^8															$h_0^7h_4$	8
7	h_0^7											$h_0^2P^1h_2$				$h_0^6h_4$	7
6	h_0^6										$h_1P^1h_1$	$h_0P^1h_2$			$h_0^2d_0$	$h_0^5h_4$	6
5	h_0^5									P^1h_1		P^1h_2			h_0d_0	h_1d_0 $h_0^4h_4$	5
4	h_0^4							$h_0^3h_3$		h_1c_0					d_0	$h_0^3h_4$	4
3	h_0^3			$h_0^2h_2$				$h_0^2h_3$	c_0	$h_1^2h_3$					$h_0h_3^2$	$h_0^2h_4$	3
2	h_0^2		h_1^2	h_0h_2			h_2^2	h_0h_3	h_1h_3						h_3^2	h_0h_4	2
1	h_0	h_1		h_2				h_3								h_4	1

TABLE II, Section 1

23														$h_0^2P^5h_2$		23
22													$h_1P^5h_1$	$h_0P^5h_2$		22
21												P^5h_1		P^5h_2		21
20										$h_0^5P^2i$		$h_1P^4c_0$				20
19						$h_0^2P^4h_2$				$h_0^4P^2i$	P^4c_0	$h_0^3P^3e_0$				19
18					$h_1P^4h_1$	$h_0P^4h_2$			$h_0^2P^3d_0$	$h_0^3P^2i$	$h_1^2P^3d_0$	$h_0^2P^3e_0$			$h_0^2P^3g$	18
17				P^4h_1		P^4h_2			$h_0P^3d_0$	$h_1P^3d_0$ $h_0^2P_2i$		$h_0P^3e_0$	$h_0^2P^2j$		h_0P^3g	17
16		$h_0^{15}h_5$		$h_1P^3c_0$					P^3d_0	h_0P^2i		P^3e_0	h_0P^2j		P^3g	16
15		$h_0^{14}h_5$	P^3c_0	$h_0^3P^2e_0$						P^2i			P^2j			15
14	$h_0^2P^2d_0$	$h_0^{13}h_5$	$h_1^2P^2d_0$	$h_0^2P^2e_0$			$h_0^2P^2g$			$h_0^2P^1d_0e_0$			$h_0^2e_0P^1e_0$			14
13	$h_0P^2d_0$	$h_1P^2d_0$ $h_0^{12}h_5$		$h_0P^2e_0$	$h_0^2P^1j$		h_0P^2g	$h_0^2P^1k$		$h_0P^1d_0e_0$	$h_0^2P^1l$		$h_0e_0P^1e_0$	$h_0^2P^1m$		13
12	P^2d_0	$h_0^{11}h_5$		P^2e_0	h_0P^1j		P^2g	h_0P^1k		$P^1d_0e_0$	h_0P^1l		$e_0P^1e_0$	h_0P^1m		12
11	h_0^5r	$h_0^{10}h_5$			P^1j			P^1k			P^1l	h_0z		P^1m		11
10	h_0^4r	$h_0^9h_5$ $h_0^2d_0e_0$			$h_0^2d_0g$			h_0^5x			h_1u	z			d_0r	10

TABLE II, Section 2

	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	
9	h_0^3r	$h_0^8h_5$ $h_0d_0e_0$	h_0^2l		h_0d_0g	h_0^2m		h_0^4x	h_0^3y	u			v			9
8	h_0^2r	$h_0^7h_5$ $h_0^7h_5+d_0e_0$	h_0l		d_0g	h_0m		h_0^3x $e_0g+h_0^3x$	h_0^2y		g^2		$h_0^2P^1h_2h_5$			8
7	h_0r	$h_0^6h_5$	l	h_1q		m		h_1t h_0^2x	h_0y	h_1y		$h_1P^1h_1h_5$	$h_0P^1h_2h_5$			7
6	r	$h_0^5h_5$	q		h_2n		t	h_0x	h_1x y		$h_0^2f_1$ $P^1h_1h_5$		$P^1h_2h_5$		$h_0^2g_1$	6
5	$h_0^3h_4^2$	n $h_0^4h_5$		h_0p		h_2d_1		x	$h_0^3h_3h_5$	h_1e_1	h_0f_1 $h_1h_5c_0$	$h_0^2c_2$			h_0g_1	5
4	$h_0^2h_4^2$	$h_0^3h_5$	d_1	p	$h_0^2h_2h_5$				e_1 $h_0^2h_3h_5$	h_5c_0	f_1 $h_1^2h_3h_5$	h_0c_2			g_1	4
3	$h_0h_4^2$	$h_1h_4^2$ $h_0^2h_5$		$k_1^2h_5$	$h_0h_2h_5$			$h_2^2h_5$	$h_0h_3h_5$	$h_1h_3h_5$		c_2				3
2	h_4^2	h_0h_5	h_1h_5		h_2h_5				h_3h_5							2
1		h_5														1
	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	

TABLE II, Section 3

TABLE III. The Adams spectral sequence for P^∞.

8															$\hat{h}_4 h_0^7$	8
7											$P^1\hat{h}_2 h_0^2$				$\hat{h}_4 h_0^6$	7
6										$\hat{h}_1 P^1 h_1$	$P^1\hat{h}_2 h_0$			$\hat{d}_0 h_0^2$	$\hat{h}_4 h_0^5$	6
5									$P^1\hat{h}_1$		$P^1\hat{h}_2$			$\hat{d}_0 h_0$	$\hat{h}_1 d_0$ $\hat{h}_4 h_0^4$	5
4							$\hat{h}_3 h_0^3$		$\hat{h}_1 c_0$	$\hat{h}_2 h_0^2 h_3$				$\hat{d}_0$	$\hat{h}_4 h_0^3$	4
3			$\hat{h}_2 h_0^2$				$\hat{h}_1 h_2^2$ $\hat{h}_3 h_0^2$	$\hat{c}_0$	$\hat{h}_1 h_1 h_3$ $\hat{h}_2 h_2^2$	$\hat{h}_2 h_0 h_3$				$\hat{h}_3 h_0 h_3$	$\hat{h}_1 h_3^2$ $\hat{h}_4 h_0^2$	3
2		$\hat{h}_1 h_1$	$\hat{h}_2 h_0$	$\hat{h}_1 h_2$		$\hat{h}_2 h_2$	$\hat{h}_3 h_0$	$\hat{h}_1 h_3$ $\hat{h}_3 h_1$		$\hat{h}_2 h_3$				$\hat{h}_3 h_3$	$\hat{h}_4 h_0$	2
1	$\hat{h}_1$		$\hat{h}_2$				$\hat{h}_3$								$\hat{h}_4$	1
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	

TABLE III, Section 1

15												$P^3\hat{h}_2h_0^2$				15
14											$\hat{h}_1P^3h_1$	$P^3\hat{h}_2h_0$			$P^2\hat{d}_0h_0^2$	14
13										$P^3\hat{h}_1$		$P^3\hat{h}_2$			$P^2\hat{d}_0h_0$	13
12								$\hat{i}h_0^5$		$\hat{h}_1P^2c_0$					$P^2\hat{d}_0$	12
11				$P^2\hat{h}_2h_0^2$				$\hat{i}h_0^4$	$P^2\hat{c}_0$	$P^1\hat{e}_0h_0^3$					$\hat{r}h_0^5$	11
10			$\hat{h}_1P^2h_1$	$P^2\hat{h}_2h_0$			$P^1\hat{d}_0h_0^2$	$\hat{i}h_0^3$	$\hat{h}_1h_1P^1d_0$	$P^1\hat{e}_0h_0^2$			$P^1\hat{g}h_0^2$		$\hat{r}h_0^4$	10
9		$P^2\hat{h}_1$		$P^2\hat{h}_2$			$P^1\hat{d}_0h_0$	$\hat{h}_1P^1d_0$ $\hat{i}h_0^2$		$P^1\hat{e}_0h_0$	$\hat{j}h_0^2$		$P^1\hat{g}h_0$	$\hat{k}h_0^2$	$\hat{r}h_0^3$	9
8		$\hat{h}_1P^1c_0$					$P^1\hat{d}_0$	$\hat{i}h_0$		$P^1\hat{e}_0$	$\hat{j}h_0$		$P^1\hat{g}$	$\hat{k}h_0$	$\hat{r}h_0^2$	8
7	$P^1\hat{c}_0$	$\hat{e}_0h_0^3$						$\hat{i}$			$\hat{j}$			$\hat{k}$	$\hat{r}h_0$	7
6	$\hat{h}_1h_1d_0$	$\hat{e}_0h_0^2$			$\hat{g}h_0^2$		$\hat{h}_1h_1g$	$\hat{h}_2h_0g$	$\hat{h}_1h_2g$		$\hat{h}_2h_2g$				$\hat{r}$	6
5		$\hat{e}_0h_0$	$\hat{f}_0h_0$	$\hat{h}_1f_0$	$\hat{g}h_0$	$\hat{h}_1g$ $\hat{h}_2f_0$	$\hat{h}_3h_0^3h_4$	$\hat{h}_1h_2c_1$ $\hat{h}_2g$	$\hat{h}_1h_4c_0$ $\hat{h}_4h_1c_0$						δ_{30} $\hat{h}_4h_0^3h_4$	5
4	α_{16}	$\hat{e}_0$	$\hat{h}_2h_0^2h_4$ $\hat{f}_0$ $\hat{h}_4h_0^2h_2$		$\hat{h}_1c_1$ $\hat{g}$	α_{21}	$\hat{h}_1h_3^3$ $\hat{h}_2c_1$ $\hat{h}_3h_0^2h_4$	$\hat{c}_0h_4$ $\hat{h}_4c_0$	$\hat{h}_2h_3^3$						$\hat{h}_4h_0^2h_4$	4
3		$\hat{h}_1h_1h_4$ $\hat{h}_2h_3^2$ $\hat{h}_4h_1^2$	$\hat{h}_2h_0h_4$ $\hat{h}_4h_0h_2$	$\hat{h}_1h_2h_4$ $\hat{c}_1$		$\hat{h}_2h_2h_4$ $\hat{h}_3h_3^2$	$\hat{h}_3h_0h_4$	$\hat{h}_3h_1h_4$							$\hat{h}_4h_0h_4$	3
2	$\hat{h}_1h_4$ $\hat{h}_4h_1$		$\hat{h}_2h_4$ $\hat{h}_4h_2$				$\hat{h}_3h_4$								$\hat{h}_4h_4$	2
	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	

TABLE III, Section 2

23														$P^5\hat{h}_2h_0^2$		23
22													$\hat{h}_1P^5h_1$	$P^5\hat{h}_2h_0$		22
21												$P^5\hat{h}_1$		$P^5\hat{h}_2$		21
20										$P^2\hat{i}h_0^5$		$\hat{h}_1P^4c_0$				20
19						$P^4\hat{h}_2h_0^2$				$P^2\hat{i}h_0^4$	$P^4\hat{c}_0$	$P^3\hat{e}_0h_0^3$				19
18					$\hat{h}_1P^4h_1$	$P^4\hat{h}_2h_0$			$P^3\hat{d}_0h_0^2$	$P^2\hat{i}h_0^3$	$\hat{h}_1h_1P^3d_0$	$P^3\hat{e}_0h_0^2$			$P^3\hat{g}h_0^2$	18
17				$P^4\hat{h}_1$		$P^4\hat{h}_2$			$P^3\hat{d}_0h_0$	$\hat{h}_1P^3d_0$ $P^2\hat{i}h_0^2$		$P^3\hat{e}_0h_0$	$P^2\hat{j}h_0^2$		$P^3\hat{g}h_0$	17
16		$\hat{h}_5h_0^{15}$		$\hat{h}_1P^3c_0$					$P^3\hat{d}_0$	$P^2\hat{i}h_0$		$P^3\hat{e}_0$	$P^2\hat{j}h_0$		$P^3\hat{g}$	16
15		$\hat{h}_5h_0^{14}$	$P^3\hat{c}_0$	$P^2\hat{e}_0h_0^3$						$P^2\hat{i}$			$P^2\hat{j}$			15
14	$P^2\hat{d}_0h_0^2$	$\hat{h}_5h_0^{13}$	$\hat{h}_1h_1P^2d_0$	$P^2\hat{e}_0h_0^2$			$P^2\hat{g}h_0^2$			$\hat{d}_0h_0^2P^1e_0$			$\hat{d}_0h_0^2P^1g$			14
13	$P^2\hat{d}_0h_0$	$\hat{h}_1P^2d_0$ $\hat{h}_5h_0^{12}$		$P^2\hat{e}_0h_0$	$P^1\hat{j}h_0^2$		$P^2\hat{g}h_0$	$P^1\hat{k}h_0^2$		$\hat{d}_0h_0P^1e_0$	$P^1\hat{l}h_0^2$		$\hat{d}_0h_0P^1g$	$P^1\hat{m}h_0^2$		13
12	$P^2\hat{d}_0$	$\hat{h}_5h_0^{11}$		$P^2\hat{e}_0$	$P^1\hat{j}h_0$		$P^2\hat{g}$	$P^1\hat{k}h_0$		$\hat{d}_0P^1e_0$	$P^1\hat{l}h_0$		$\hat{d}_0P^1g$	$P^1\hat{m}h_0$		12
11	$\hat{r}h_0^5$	$\hat{h}_5h_0^{10}$			$P^1\hat{j}$			$P^1\hat{k}$			$P^1\hat{l}$	$\hat{z}h_0$		$P^1\hat{m}$		11
10	$\hat{r}h_0^4$	$\hat{d}_0h_0^2e_0$ $\hat{h}_5h_0^9$			$\hat{d}_0h_0^2g$			$\hat{x}h_0^5$			$\hat{h}_1u$	$\hat{z}$			$\hat{d}_0r$	10
9	$\hat{r}h_0^3$	$\hat{d}_0h_0e_0$ $\hat{h}_5h_0^8$	$\hat{l}h_0^2$		$\hat{d}_0h_0g$	$\hat{m}h_0^2$		$\hat{x}h_0^4$	$\hat{y}h_0^3$	$\hat{u}$		$\hat{h}_1g^2$	$\hat{v}$			9

TABLE III, Section 3

	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	
8	$\hat{r}h_0^2$	$\hat{d}_0e_0+\hat{h}_5h_0^7$ $\hat{h}_5h_0^7$	$\hat{l}h_0$		$\beta_{34}h_0^2$ $\hat{d}_0g$	$\hat{m}h_0$	$\hat{h}_1m$	$\hat{e}_0g+\hat{x}h_0^3$ $\hat{x}h_0^3$	$\beta_{35}h_0h_2$ $\hat{y}h_0^2$		$\hat{h}_1h_1y$ $\hat{g}g$	$\hat{h}_2h_0y$	$P^1\hat{h}_2h_0^2h_5$ $\hat{h}_5h_0^2P^1h_2$			8
7	$\hat{r}h_0$	$\hat{h}_1r$ $\hat{h}_5h_0^6$	$\hat{l}$	$\hat{h}_1q$ $\hat{h}_2r$	$\beta_{34}h_0$	$\beta_{35}h_0$ $\hat{m}$	$\delta_{30}h_2^2$	$\hat{h}_1t$ $\hat{h}_2h_2n$ $\hat{x}h_0^2$	$\beta_{35}h_2$ $\hat{y}h_0$	$\hat{h}_1h_1x$ $\hat{h}_1y$ $\hat{h}_2t$		$\hat{h}_1P^1h_1h_5$ $\hat{h}_2y$ $\hat{h}_5P^1h_1^2$	$P^1\hat{h}_2h_0h_5$ $\hat{h}_5P^1h_0h_2$		$\gamma_{37}h_0h_3$	7
6	$\hat{r}$	$\hat{h}_5h_0^5$	$\hat{h}_1n$ α_{32} $\hat{q}$	$\delta_{30}h_2$	$\hat{h}_2n$ β_{34}	β_{35}	$\hat{h}_1h_2d_1$ $\hat{t}$	$\zeta_{31}h_2^2$ $\gamma_{37}h_0$ $\hat{x}h_0$	$\hat{h}_1x$ $\hat{h}_2h_2d_1$ $\hat{y}$	$\zeta_{31}h_1h_3$	$\alpha_{40}h_0$ $P^1\hat{h}_1h_5$ $\hat{f}_1h_0^2$ $\hat{h}_5P^1h_1$		$P^1\hat{h}_2h_5$ $\hat{h}_5P^1h_2$		$\gamma_{37}h_3$ $\hat{g}_1h_0^2$	6
5	δ_{30} $\hat{h}_4h_0^3h_4$	$\hat{n}$ $\hat{h}_5h_0^4$	$\zeta_{31}h_1$	$\hat{h}_1d_1$ $\hat{p}h_0$	$\zeta_{31}h_2$	$\hat{h}_2d_1$		γ_{37} $\hat{x}$	$\hat{h}_3h_0^3h_5$ $\zeta_{31}h_3$ $\hat{h}_5h_0^3h_3$	$\hat{h}_1e_1$ $\hat{h}_3d_1$	$\hat{h}_1h_5c_0$ α_{40} $\hat{f}_1h_0$ $\hat{h}_5h_1c_0$	$\hat{h}_1f_1$ $\hat{h}_2h_0^2h_3h_5$ $\hat{c}_2h_0^2$		$\hat{h}_2f_1$	$\hat{g}_1h_0$	5
4	$\hat{h}_4h_0^2h_4$	ζ_{31} $\hat{h}_5h_0^3$	$\hat{h}_1h_1h_4^2$ $\hat{d}_1$	$\hat{h}_2h_0h_4^2$ $\hat{p}$	$\hat{h}_2h_0^2h_5$ $\hat{h}_5h_0^2h_2$		γ_{36}	$\hat{h}_3h_0h_4^2$	$\hat{h}_1h_2^2h_5$ $\hat{e}_1$ $\hat{h}_3h_0^2h_5$ $\hat{h}_5h_0^2h_3$	$\hat{c}_0h_5$ $\hat{h}_5c_0$	$\hat{h}_1h_1h_3h_5$ $\hat{h}_2h_2^2h_5$ $\hat{f}_1$ $\hat{h}_5h_2^3$	$\hat{h}_2h_0h_3h_5$ $\hat{c}_2h_0$	$\hat{h}_1c_2$		$\hat{h}_2c_2$ $\hat{g}_1$	4
3	$\hat{h}_4h_0h_4$	$\hat{h}_1h_4^2$ $\hat{h}_4h_1h_4$ $\hat{h}_5h_0^2$		$\hat{h}_1h_1h_5$ $\hat{h}_2h_4^2$ $\hat{h}_5h_1^2$	$\hat{h}_2h_0h_5$ $\hat{h}_5h_0h_2$	$\hat{h}_1h_2h_5$		$\hat{h}_2h_2h_5$ $\hat{h}_3h_4^2$ $\hat{h}_5h_2^2$	$\hat{h}_3h_0h_5$ $\hat{h}_5h_0h_3$	$\hat{h}_1h_3h_5$ $\hat{h}_3h_1h_5$ $\hat{h}_5h_1h_3$		$\hat{h}_2h_3h_5$ $\hat{c}_2$				3
2	$\hat{h}_4h_4$	$\hat{h}_5h_0$	$\hat{h}_1h_5$ $\hat{h}_5h_1$		$\hat{h}_2h_5$ $\hat{h}_5h_2$				$\hat{h}_3h_5$ $\hat{h}_5h_3$							2
1		$\hat{h}_5$														1

TABLE III, Section 4

TABLE IV. Stable homotopy groups of S^0 and P^∞.

n	$\sigma_n(2)$	$\sigma_n(P^\infty)$	Ker j
1	$Z_2(\eta)$	$Z_2(\hat{\eta})$	0
2	$Z_2(\eta^2)$	$Z_2(\hat{\eta}\eta)$	0
3	$Z_8(\nu)$	$Z_8(\hat{\nu})$	0
4	0	$Z_2(\hat{\eta}\nu)$	$Z_2(\hat{\eta}\nu)$
5	0	0	0
6	$Z_2(\nu^2)$	$Z_2(\hat{\nu}\nu)$	0
7	$Z_{16}(\sigma)$	$Z_2(\hat{\eta}\,\nu^2) + Z_{16}(\hat{\sigma})$	$Z_2(\hat{\eta}\nu^2)$
8	$Z_2(\eta\sigma) + Z_2(\epsilon)$	$Z_2(\hat{\eta}\sigma) + Z_2(\hat{\sigma}\eta) + Z_2(\hat{\epsilon})$	$Z_2(\hat{\eta}\sigma + \hat{\sigma}\eta)$
9	$Z_2(\nu^3) + Z_2(\eta\epsilon) + Z_2(\mu)$	$Z_2(\hat{\eta}\eta\sigma) + Z_2(\hat{\nu}\nu^2) + Z_2(\hat{\eta}\epsilon) + Z_2(\hat{\mu})$	$Z_2((\hat{\eta}\sigma + \hat{\sigma}\eta)\eta)$
10	$Z_2(\eta\mu)$	$Z_2(\hat{\eta}\mu) + Z_8(\hat{\nu}\sigma)$	$Z_8(\hat{\nu}\sigma)$
11	$Z_8(\zeta)$	$Z_8(\hat{\zeta})$	0
12/13	0	0	0
14	$Z_2(\sigma^2) + Z_2(\kappa)$	$Z_2(\hat{\sigma}\sigma) + Z_2(\hat{\kappa})$	0
15	$Z_{32}(\rho) + Z_2(\eta\kappa)$	$Z_{32}(\hat{\rho}) + Z_2(\hat{\eta}\kappa) + Z_2(\hat{\eta}\sigma^2)$	$Z_2(\hat{\eta}\sigma^2)$
16	$Z_2(\eta\rho) + Z_2(\eta^*)$	$Z_2(\hat{\eta}_1^*) + Z_2(\hat{\eta}_2^*) + Z_2([\alpha_{16}]) + Z_2(\hat{\eta}\rho)$	$Z_2(\hat{\eta}_1^* + \hat{\eta}_2^*) + Z_2([\alpha_{16}])$
17	$Z_2(\eta\eta^*) + Z_2(\nu\kappa)$ $+ Z_2(\eta^2\rho) + Z_2(\bar{\mu})$	$Z_2(\hat{\eta}_1^*\eta) + Z_2(\hat{\eta}\eta^*) + Z_2(\hat{\nu}\sigma^2)$ $+ Z_2(\hat{\nu}\kappa) + Z_2(\hat{\eta}\eta\rho) + Z_2(\hat{\bar{\mu}})$	$Z_2((\hat{\eta}_1^* + \hat{\eta}_2^*)\eta) + Z_2(\hat{\nu}\sigma^2)$
18	$Z_8(\nu^*) + Z_2(\eta\bar{\mu})$	$Z_8(\hat{\nu}_1^*) + Z_8(\hat{\nu}_2^*) + Z_2(\hat{\eta}\bar{\mu})$	$Z_8((\hat{\nu}_1^* - \hat{\nu}_2^*))$
19	$Z_2(\bar{\sigma}) + Z_8(\bar{\zeta})$	$Z_2(\hat{\bar{\sigma}}) + Z_2(\hat{\eta}\nu^*) + Z_2([\alpha_{16}]\nu) + Z_8(\hat{\bar{\zeta}})$	$Z_2(\hat{\eta}\nu^*) + Z_2([\alpha_{16}]\nu)$
20	$Z_8(\bar{\kappa})$	$Z_2(\hat{\eta}\bar{\sigma}) + Z_8(\hat{\bar{\kappa}})$	$Z_2(\hat{\eta}\bar{\sigma})$
21	$Z_2(\sigma^3) + Z_2(\eta\bar{\kappa})$	$Z_2(\hat{\nu}\nu^*) + Z_2(\hat{\sigma}\sigma^2) + Z_2([\alpha_{21}])$ $+ Z_2(\hat{\eta}\bar{\kappa}) + Z_2(\hat{\bar{\kappa}}\eta)$	$Z_2(\hat{\nu}\nu^* + \hat{\sigma}\sigma^2) + Z_2([\alpha_{21}])$ $+ Z_2(\hat{\eta}\bar{\kappa} + \hat{\bar{\kappa}}\eta)$
22	$Z_2(\nu\bar{\sigma}) + Z_2(\epsilon\kappa)$	$Z_{16}(\hat{\sigma}^*) + Z_2(\hat{\nu}\bar{\sigma}) + Z_2(\hat{\eta}\sigma^3)$ $+ Z_2(\hat{\eta}\eta\bar{\kappa}) + Z_2(\hat{\epsilon}\kappa)$	$Z_{16}(\hat{\sigma}^*) + Z_2(\hat{\eta}\sigma^3)$ $+ Z_2((\hat{\eta}\bar{\kappa} + \hat{\bar{\kappa}}\eta)\eta)$
23	$Z_{16}(\bar{\rho}) + Z_8(\nu\bar{\kappa}) + Z_2(\eta^*\sigma)$	$Z_2(\hat{\sigma}\eta^*) + Z_2(\hat{\eta}_1^*\sigma) + Z_2(\hat{\eta}_2^*\sigma) + Z_2(\hat{\eta}\nu\bar{\sigma})$ $+ Z_8(\hat{\nu}\bar{\kappa}) + Z_{16}(\hat{\bar{\rho}})$	$Z_2(\hat{\sigma}\eta^* + \hat{\eta}_1^*\sigma) + Z_2((\hat{\eta}_1^* + \hat{\eta}_2^*)\sigma)$ $+ Z_2(\hat{\eta}\nu\bar{\sigma})$
24	$Z_2(\eta\eta^*\sigma) + Z_2(\eta\bar{\rho})$	$Z_2(\hat{\nu}\sigma^3) + Z_2(\hat{\eta}\eta^*\sigma) + Z_2(\hat{\eta}_2^*\eta\sigma) + Z_2(\hat{\eta}\nu\bar{\kappa}) + Z_2(\hat{\eta}\bar{\rho})$	$Z_2(\hat{\nu}\sigma^3) + Z_2((\hat{\eta}\eta^* + \hat{\eta}_2^*\eta)\sigma) + Z_2(\hat{\eta}\nu\bar{\kappa})$
25	$Z_2(\eta^2\bar{\rho}) + Z_2(\bar{\bar{\mu}})$	$Z_2(\hat{\eta}\eta\bar{\rho}) + Z_2(\hat{\bar{\bar{\mu}}})$	0
26	$Z_2(\nu^2\bar{\kappa}) + Z_2(\eta\bar{\bar{\mu}})$	$Z_2(\hat{\nu}\nu\bar{\kappa}) + Z_2(\hat{\eta}\bar{\bar{\mu}})$	0
27	$Z_8(\bar{\bar{\zeta}})$	$Z_8(\hat{\bar{\bar{\zeta}}})$	0
28	$Z_2(\epsilon\bar{\kappa})$	$Z_2(\hat{\epsilon}\bar{\kappa})$	0
29	0	0	0
30	$Z_2(\theta_4)$	$Z_2(\hat{\theta}_4) + Z_2([\delta_{30}])$	$Z_2([\delta_{30}])$

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