Conference Board of the Mathematical Sciences
REGIONAL CONFERENCE SERIES IN MATHEMATICS

Number 11

Irving Glicksberg

Recent results on function algebras

Supported by the National Science Foundation
Published by the American Mathematical Society

http://dx.doi.org/10.1090/cbms/011

Conference Board of the Mathematical Sciences

REGIONAL CONFERENCE SERIES IN MATHEMATICS

supported by the

National Science Foundation

Number 11

RECENT RESULTS ON FUNCTION ALGEBRAS

by

IRVING GLICKSBERG

Published for the

Conference Board of the Mathematical Sciences

by the

American Mathematical Society

Providence, Rhode Island

Expository Lectures
from the CBMS Regional Conference
held at Fayetteville, Arkansas
June 1971

International Standard Book Number 0-8218-9901-5
Library of Congress Catalog Card Number 75-38927
AMS 1969 Subject Classifications: 3070, 4655, 4687
AMS On Demand ISBN 0-8218-3878-4

Printed in the United States of America

FOREWORD

What follows are notes covering lectures given at a Regional Conference at Fayetteville, Arkansas in June 1971, sponsored by CBMS and the NSF. My aim in these lectures was to introduce some of the more recent developments on function algebras, presuming no detailed knowledge on the listener's part. Of course then it was necessary to develop some of the foundations, and some older results are interspersed among the newer ones. Some areas are also slighted considerably: for example, we shall have nothing to say about analytic structure in the spectrum; and because of the technical aspect, we will really cover only one result in rational approximation in the plane (an area where considerable recent progress has been made via a combination of the classical and abstract techniques), and will wholly ignore higher dimensions.

Here, briefly, is what we attempt to cover: (1) some basic notions, results and examples of uniform algebras, (2) interpolation, (3) orthogonal measures, (4) rational approximation, and (5) recent characterizations of $C(X)$.

I am greatly indebted to various friends for kindly providing preprints and the latest and simplest versions of proofs, particularly A. Bernard, B. Cole, A. M. Davie, T. W. Gamelin, J. Garnett, S. J. Sidney, and N. T. Varopoulos. Finally, thanks are due the Department of Mathematics of the University of Arkansas, and especially Professor Allan Cochran, without whose untiring efforts the conference could not have taken place.

http://dx.doi.org/10.1090/cbms/011/05

choose a high power of the peaking function as a $g' \in A_R^-$ with $\|g'g\| < 1 + \epsilon$, $g' \equiv 1$ on K. For an approximating g_0 in A_R with $g_0(x_2) = 1 = g'(x_2)$ we will also have $\|g_0 g\| < 1 + \epsilon$, and so for $h = g_0 g \in A$, $h(x_1) = 0$, $h(x_2) = 1$, $\|h\| < 1 + \epsilon$.

Now with $\epsilon > 0$ sufficiently small we can find a polynomial p which maps the disc $(1 + \epsilon)D$ into $|\mathrm{Re}\, z| < 1$ and has $p(0) = 0, p(1) = (2k + 1)i$. (We can choose a conformal map ϕ of D° onto $|\mathrm{Re}\, z| < 1$ which is 0 at 0, imaginary on $R \cap D$, and has $\mathrm{Im}\, \phi(z) \longrightarrow +\infty$ for $z \longrightarrow 1$ so that for ϵ small $z \longrightarrow \phi(z/(1 + 2\epsilon))$ has a large imaginary value at $z = 1$, approximate this on $(1 + \epsilon)D$ by a polynomial still 0 at 0 and imaginary on $R \cap D$, and take p as a real multiple.) Thus $f = p \circ h \in A$ will yield our assertion.

Now if A_R does not separate X we have such a K, x_1, x_2 and f. By (5.1) for $u = \mathrm{Re}\, f$ we have a $v \in C_R(X)$ with $u + iv \in A$, $N(u + iv) \leqslant k$, so $\|u + iv\| \leqslant k$, and $a = (1/i)\,(f - (u + iv)) \in A_R$ while

$$|a(x_1)| = |v(x_1)| \leqslant \|v\| \leqslant \|u + iv\| \leqslant k$$

and

$$|a(x_2)| \geqslant 2k + 1 - |v(x_2)| \geqslant k + 1$$

so that $a(x_1) \neq a(x_2)$ despite the fact that $x_1, x_2 \in K$. Our proof of 5.2 is complete.

The proof of the Hoffman-Wermer Theorem 5.1 is now immediate: Since A_R is dense in $C_R(X)$ by Stone-Weierstrass, and closed since $A_R = A \cap C_R(X)$ and A is, $A_R = C_R(X)$. But of course that says $A \supset A_R + iA_R = C(X)$.

As a corollary to the Hoffman-Wermer theorem Bernard obtained a far reaching generalization.

THEOREM 5.3. *Suppose* $1 \in A$, *a Banach subalgebra of* $C(X)$ *which separates* X. *Then* $\mathrm{Re}\, A$ *is uniformly closed only if* $A = C(X)$.

This extends (and no doubt was suggested by) an earlier result of Sidney and Stout [S–S], that for a separating uniform algebra A, $\mathrm{Re}\, A\,|F$ is closed in $C(F)$ only if $A|F = C(F)$ (take $A|F$ as the Banach subalgebra of $C(F)$, normed of course as the isomorphic quotient algebra A/kF). The fundamental step in 5.3 is the remarkably simple and powerful

LEMMA 5.4 (BERNARD'S LEMMA). *Suppose* $E \subset F$, *each normed spaces, with the injection continuous. Let* $\widetilde{E} = l_\infty(N, E)$ *(the bounded functions from* $N = \{1, 2, \cdots\}$ *to* E, *normed as usual),* $\widetilde{F} = l_\infty(N, F)$. *Then* $\widetilde{E} \subset \widetilde{F}$, *and if* E *is a Banach space,* $\widetilde{E}$ *dense in* $\widetilde{F}$ *implies* $E = F$.

That $\widetilde{E} \subset \widetilde{F}$ is trivial since $E \xrightarrow{\subset} F$ is continuous. From the final portion of the usual proof of the open mapping theorem one can observe that to see $E = F$ it suffices to see that for some n, nB_E, the n-ball in E, is ½-dense in B_F (i. e., for $y \in B_F$ there is an $x \in nB_E$ with $\|x - y\|_F < ½$). And if that fails, for each n we have a y_n in B_F with $\|y_n - nB_E\|_F \geqslant ½$. But by hypothesis there is an $x = \{x_n\}$ in $\widetilde{E}$ with $\|x_n - y_n\| < ¼$ for all n, and so $x_n \notin nB_E$ and $\|x_n\|_E > n$ for each n, so that $x \notin \widetilde{E}$.

Bernard's proof of Theorem 5.3 is now as follows: since $\mathrm{Re}\, A$ is closed, $\mathrm{Re}\, A^- \subset \mathrm{Re}\, A$, where A^- is the uniform closure of A, so $\mathrm{Re}\, A^- = \mathrm{Re}\, A$ is closed. Thus the

http://dx.doi.org/10.1090/cbms/011/05

Hoffman-Wermer theorem says $A^- = C(X)$, whence $\operatorname{Re} A^- = \operatorname{Re} A = C_R(X)$. So by the open mapping theorem we have a k for which

(5.3) $\qquad u \in C_R(X)$ implies $u + iv \in A$, $v \in C_R(X)$, $N(u + iv) \leqslant k\|u\|$.

Now $\widetilde{A} = l_\infty(N, A)$ is a subalgebra of $l_\infty(N, C(X))$, and we can identify the latter with $C(\beta(N \times X))$, so (5.3) allows us to assert that

$$\operatorname{Re} \widetilde{A} = C_R(\beta(N \times X)): \tag{5.4}$$

for with $u = \{u_n\} \in \operatorname{Re} l_\infty(N, C(X)) = l_\infty(N, C_R(X))$ we have $v_n \in C_R(X), N(u_n + iv_n) \leqslant k\|u\|_\infty = k \sup \|u_n\|$, $u_n + iv_n \in A$, so that $\widetilde{a} = \{u_n + iv_n\} \in \widetilde{A}$ and $\operatorname{Re} \widetilde{a} = u$.

Since (5.4) clearly implies $\widetilde{A}$ separates $\beta(N \times X)$, Hoffman-Wermer applies to $\widetilde{A}^-$ (since $\operatorname{Re} \widetilde{A}^- = \operatorname{Re} \widetilde{A} = C_R(\beta(N \times X))$). Thus $\widetilde{A}^- = C(\beta(N \times X)) = l_\infty(N, C(X))$ and that says $A = C(X)$ by Bernard's lemma.

One application of interest occurs in harmonic analysis. If Γ is a compact abelian group, the dual of a discrete group G, a closed subset H of Γ is called a Helson set if $l_1(G)^\wedge \mid H = C(H)$. Now if S is a subsemigroup of G with $S \cup (-S) = G$ then $l_1(S)^\wedge | H = C(H)$ for any Helson set H: for $\operatorname{Re} (l_1(S)^\wedge | H) = \operatorname{Re} (l_1(G)^\wedge | H) = \operatorname{Re} C(H) = C_R(H)$, as is easily seen, so $l_1(S)^\wedge | H = C(H)$ by Theorem 5.3. (A special case is known as Wik's theorem.)

Bernard has exploited the kind of argument used to prove Theorem 5.3, as well as the result itself, to obtain a variety of consequences, in particular an extension of Wermer's theorem that if A is uniformly closed and separating $\operatorname{Re} A$ cannot be an algebra unless $A = C(X)$, and, recently, that $\operatorname{Re} A$ cannot be a lattice. In this the following notion is useful.

DEFINITION. *$A \subset C(X)$ is called ultraseparating if $\widetilde{A}$ separates $\beta(N \times X)$ (and ultraseparating on $K \subset X$ if $\widetilde{(A|K)}$ separates $\beta(N \times K)$).*

In particular both occur if a uniform algebra A "approximates in modulus", i.e., given $\epsilon > 0$ and $f > 0$ in $C_R(X)$ there is an $a \in A$ with $|f - |a|| < \epsilon$: for if $f \in C_R(\beta(N \times X))$ is zero at one point x^* of $\beta(N \times X)$ and 1 at another, y^*, $0 \leqslant f \leqslant 1$, then as an element of $l_\infty(N, C_R(X))$ $f = \{f_n\}$, and we can find a_n in A with $|f_n - |a_n|| < 1/3$ so that $\widetilde{a} = \{a_n\} \in \widetilde{A}$, and $\widetilde{a}$ (extended to $\beta(N \times X)$) has a value of modulus $\geqslant 2/3$ at y^*, $\leqslant 1/3$ at x^*. Finally, it is easy to see ultraseparation implies ultraseparation on closed subsets.

LEMMA 5.5. *If $1 \in A$, a Banach subalgebra of $C(X)$, A is ultraseparating and $B \supset A$ is another Banach subalgebra of $C(X)$ which is conjugate closed, then $B = C(X)$.*

Since the point evaluations $a \longrightarrow a(x)$ and $b \longrightarrow b(x)$ are continuous on A and B the closed graph theorem shows the inclusion $A \xrightarrow{\subset} B$ is continuous. As a consequence $\widetilde{A} \subset \widetilde{B} (\subset l_\infty(N, C(X)))$ and thus $\widetilde{B}$ separates $\beta(N \times X)$. Moreover the closed graph theorem shows conjugation is continuous on B, and that implies $\widetilde{B}$ is conjugate closed (in $l_\infty(N, C(X))$, hence in $C(\beta(N \times X))$). So $\widetilde{B}$ is dense in $C(\beta(N \times X))$ by Stone-Weierstrass, or $\widetilde{B}$ is dense in $\widetilde{C(X)}$, whence $B = C(X)$ by Bernard's Lemma 5.4.

http://dx.doi.org/10.1090/cbms/011/05

THEOREM 5.6. *If* $1 \in A$, *and* A *is an ultraseparating Banach subalgebra of* $C(X)$ *and* $\operatorname{Re} A$ *is closed under multiplication then* $A = C(X)$.

We have only to take $B = \operatorname{Re} A + i \operatorname{Re} A$, which is by hypothesis a subalgebra of $C(X)$ containing A, and also a Banach space which injects continuously into $C(X)$ when normed by $N(\operatorname{Re} a + i \operatorname{Re} b) = N_1(\operatorname{Re} a) + N_1(\operatorname{Re} b)$, where $N_1(\operatorname{Re} a) = \|a + E\|$ is the quotient norm in the real Banach space A/E and E is the kernel of $a \longrightarrow \operatorname{Re} a$. By our preliminary remarks, B is a Banach subalgebra of $C(X)$ and from Lemma 5.5 we have $B = \operatorname{Re} A + i \operatorname{Re} A = C(X)$, so $\operatorname{Re} B = \operatorname{Re} A = C_R(X)$. And that implies $A = C(X)$ by Bernard's Theorem 5.3.

COROLLARY 5.7 (WERMER). *If* A *is a closed separating subalgebra of* $C(X)$ *containing the constants and* $\operatorname{Re} A$ *is an algebra then* $A = C(X)$.

For $\operatorname{Re} A$ is necessarily dense in $C_R(X)$, so that A "approximates in modulus".

COROLLARY 5.8. *If* $1 \in A$, *a closed subalgebra of* $C(X)$ *which approximates in modulus, then* $\operatorname{Re}(A|K)$ *is closed under multiplication only if* $A|K = C(K)$.

For $A|K$ is ultraseparating on K as noted earlier. For the disc algebra $A = A(D)$ this has the following consequence: if $K \subset T$ is compact and has positive measure then there is a real continuous function u on K which has an extension in $\operatorname{Re} A(D)$ while u^2 does not. (Alternatively, u has a continuous extension to D harmonic on D° which has a continuous harmonic conjugate, while u^2 does not.)

Most recently Bernard has shown $\operatorname{Re} A$ cannot be a lattice when A is a proper closed separating subalgebra of $C(X)$. This depends on the following simple result of de Leeuw and Katznelson.

THEOREM 5.9. *Suppose* E *is a closed separating subspace of* $C_R(X)$ *and* $1 \in E$. *If a non-affine continuous* $\phi : R \longrightarrow R$ *operates on* E (*i.e. if* $\phi \circ E \subset E$) *then* $E = C_R(X)$.

PROOF. Trivially the convolution $k * \phi = \phi'$ will operate as well since ϕ' is a uniform limit on compacta of linear combinations of translates of ϕ (all of which operate since $1 \in E$!) and E is uniformly closed. Taking k to be C^∞ with small support we can insure that ϕ' is close enough to ϕ to also be non-affine, and C^∞. Replacing ϕ' by $\phi_1(t) = a\phi(t - t_0) + bt + c$ for appropriate a, b, c, t_0 we can assume $\phi_1(0) = \phi_1'(0) = 0, \phi_1''(0) = 2$, so $\phi_1(t) = t^2 + t^2\epsilon(t)$, where $\epsilon(t) \longrightarrow 0$ as $t \longrightarrow 0$.

But for any integer n,

$$\phi_n(t) = n^2 \phi_1(t/n) = t^2 + t^2 \epsilon(t/n)$$

operates; so finally t^2 operates, i.e., E is a closed separating subalgebra of $C_R(X)$, $1 \in E$, so $E = C_R(X)$ by Stone-Weierstrass.

Suppose now $E \subset C_R(X)$ is normed so the injection into $C_R(X)$ is continuous. Then $\phi : R \longrightarrow R$ *operates boundedly on* E if ϕ is continuous and for each $\alpha > 0$ there is an M for which

$$N(u) < \alpha \text{ implies } \phi \circ u \in E \text{ and } N(\phi \circ u) \leqslant M. \tag{5.5}$$

http://dx.doi.org/10.1090/cbms/011/05

THEOREM 5.10. *If A is a closed proper separating subalgebra of $C(X)$ and $1 \in A$ only affine functions operate boundedly on* Re A (*taken with its natural quotient norm*).

THEOREM 5.11. *If A is a closed proper separating subalgebra of $C(X)$ and $1 \in A$,* Re A *is not a lattice.*

Theorem 5.11 will be obtained from the argument leading to Theorem 5.10 by showing that if $\phi(t) = |t|$ operates on Re A it must operate boundedly. We begin with the main connecting link to the earlier results.

LEMMA 5.12. *Suppose E is a* (*real*) *Banach space which injects continuously into $C_R(X)$ as above, contains 1 and is ultraseparating on X. If $E \neq C_R(X)$ then only affine functions operate boundedly on E.*

Trivially if ϕ operates on E then ϕ operates on E^- in $C_R(X)$: if $u_n \in E$ and $u_n \longrightarrow f$ in $C_R(X)$ then $\phi \circ u_n \in E$ and $\phi \circ u_n \longrightarrow \phi \circ f$ uniformly, so $\phi \circ f \in E^-$. Since ϕ operates boundedly on E, it operates on $\widetilde{E} = l_\infty(N, E) \subset \widetilde{C_R(X)} = C_R(\beta(N \times X))$, hence it operates on the closure F of $\widetilde{E}$ in $C_R(\beta(N \times X))$. But ϕ non-affine implies $F = C_R(\beta(N \times X))$ by Katznelson and de Leeuw's Theorem 5.9, which applies since $\widetilde{E}$ separates $\beta(N \times X)$, i.e. E is ultraseparating. Hence $\widetilde{E}$ is dense in $\widetilde{C_R(X)}$, and $E = C_R(X)$ by Bernard's lemma.

In case Re A is dense in $C_R(X)$ the lemma applies to $E =$ Re A since then Re A is ultraseparating. To prove 5.10, suppose ϕ is non-affine and operates boundedly on Re A. Then it operates on (Re $A)^-$, so since (Re $A)^-$ separates, Theorem 5.9 implies (Re $A)^- = C_R(X)$. Now Re A is ultraseparating on X so Lemma 5.12 applies to show Re $A = C_R(X)$. By Bernard's theorem, $A = C(X)$.

To prove Theorem 5.11 we need another lemma.

LEMMA 5.13. *Suppose $E \subset C_R(X)$ is a Banach space which injects continuously into $C_R(X)$, contains 1, separates X, and is a lattice. Then there is a finite subset F of X such that $x \notin F$ implies x has a compact neighborhood K_x for which $\phi(t) = |t|$ operates boundedly on $E \mid K_x$* (*taken with the natural quotient norm*).

For a compact $K \subset U$ open, an easy compactness argument (using the fact that E is a lattice which separates, and $0, 1 \in E$) shows that any element u of $E \mid K$ has an extension $\widetilde{u}$ in $E \equiv 0$ off U. (For $0 \leqslant u \leqslant 1$ it suffices to find a v in E, $= 1$ on K and $\equiv 0$ off U, and take $\widetilde{u} = u' \wedge v$, where $u' | K = u$.) By the open mapping theorem then there is a constant $k_{U,K}$ for which, for each $u \in E | K$, there is a $\widetilde{u}$ as above with

$$N(\widetilde{u}) \leqslant k_{U,K} N_K(u) \tag{5.6}$$

where $N_K(u) = \inf \{N(u') : u' = u \text{ on } K, \ u' \in E\}$.

Suppose the lemma fails and let E be the infinite set of x for which ϕ fails to operate boundedly on $E|K_x$ for each compact neighborhood K_x of x. Let x_0 be a cluster point of E. For $x_1 \in E \backslash \{x_0\}$ let K_1 be a compact neighborhood of x_1 in $E \backslash \{x_0\}$, and choose x_2 and a compact neighborhood K_2 of x_2 in $E \backslash (K_1 \cup \{x_0\})$. Continuing we obtain a sequence of disjoint compact neighborhoods K_n for which

http://dx.doi.org/10.1090/cbms/011/05

$\phi(t) = |t|$ does not operate boundedly on $E|K_n$; we now choose open $U_n \supset K_n$ so that the U_n are also pairwise disjoint.

For each n we have $u_{n,k} \in E|K_n$ with $N_{K_n}(u_{n,k}) \longrightarrow 0$ as $k \longrightarrow \infty$ while all extensions of $|u_{n,k}|$ in E have norm $\geqslant n$, and thus in view of (5.6) we have elements $\widetilde{u}_n$ of E with $\widetilde{u}_n \equiv 0$ off U_n, $N(\widetilde{u}_n) < 2^{-n}$, while each extension of $|\widetilde{u}_n| \,|K_n$ in E has norm $\geqslant n$. But consider $\widetilde{u} = \Sigma \widetilde{u}_n$. It lies in E, so $|\widetilde{u}| \in E$, while $|\widetilde{u}| \,| K_n = |\widetilde{u}_n| \,|K_n$, so that $N(|\widetilde{u}|) \geqslant n$ for all n, the desired contradiction.

We can now prove 5.11. Suppose Re A is a lattice, and thus dense in $C_R(X)$. Then Re $A|K$ is dense in $C_R(K)$, so Re $A|K$ is ultraseparating on K. Applying 5.13 to $E =$ Re A we have a finite $F \subset X$ with $\phi(t) = |t|$ operating boundedly on Re $A|K_x, x \notin F$, where K_x is a neighborhood of x. And since Theorem 5.10 applies to Re $A|K_x$ to show Re $A|K_x = C_R(K_x)$, we know $A|K_x = C(K_x)$ by Bernard's theorem, for each $x \notin F$. It is now easy to see $A = C(X)$. (Consider the essential set ! [Br, p. 144]).

For further results along these lines the reader should see [Be 5].

http://dx.doi.org/10.1090/cbms/011/05

BIBLIOGRAPHY

At the time of the conference two texts on function algebras were available:

[Br] A. Browder, *Introduction to function algebras,* Benjamin, New York, 1969. MR **39** #7431.

[Gam] T. W. Gamelin, *Uniform algebras,* Prentice-Hall, Englewood Cliffs, N. J., 1969.

Of these Browder's is the more elementary, Gamelin's the more succinct and extensive. A beginner might well approach them in sequence. Both are concerned with rational approximation, but only Gamelin covers Vituskin's work (with some minor flaws in the presentation; one can also consult Zalcman's monograph). Two more texts have since appeared:

J. Wermer, *Banach algebras and several complex variables,* Markham, Chicago, Ill., 1971.

which approaches approximation problems for subsets of $\mathbf{C}^n$, and includes results on analytic structure in spectra, and

E. L. Stout, *The theory of uniform algebras,* Bogden and Quigley, Tarrytown on Hudson, New York, 1971.

The following are referred to explicitly above

[Ba] R. F. Basener, *An example concerning peak points,* Brown University, Providence, R. I. (preprint).

[Be 1] A. Bernard, *Une caractérisation de* $C(X)$ *parmi les algèbres de Banach,* C. R. Acad. Sci. Paris Sér. A-B **267** (1968), A634–A635. MR **38** #2601.

[D] A. M. Davie, *Bounded limits of analytic functions,* University of California, Los Angeles (preprint).

[Du] A. Dufresnoy, *Parties réelles de certains quotients d'algèbres uniform,* C. R. Acad. Sci. Paris (to appear).

[Gam–Gar] T. W. Gamelin and J. Garnett, *Bounded approximation by rational functions,* University of California, Los Angeles (preprint).

[Be 2] A. Bernard, *Comparison d'algèbres de fonctions à l'aide des parties réelles de leurs éléments,* C. R. Acad. Sci. Paris Sér. A-B **270** (1970), A29–A32. MR **41** #4245.

[Be 3] ———, *Algèbres ultraséparantes de fonctions continues,* C. R. Acad. Sci. Paris Sér. A-B **270** (1970), A818–A819. MR **41** #826.

[Be 4] ———, *Fonctions qui opèrent sur* Re A, C. R. Acad. Sci. Paris Sér. A-B **271** (1970), A1120–A1121. MR **42** #5048.

[Be 5] ———, *Espace des parties réelles des éléments d'une algèbre de Banach de fonctions,* J. Functional Analysis (to appear).

http://dx.doi.org/10.1090/cbms/011/05

[D–S] N. Dunford and J. T. Schwartz, *Linear operators.* I: *General theory,* Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR **22** #8302.

[Gar–Gl] J. Garnett and I. Glicksberg, *Algebras with the same multiplicative measures,* J. Functional Analysis **1** (1967), 331–341. MR **36** #691.

[Gl] I. Glicksberg, *The abstract F. and M. Riesz theorem,* J. Functional Analysis **1** (1967), 109–122. MR **35** #2146.

[H] K. Hoffman, *Banach spaces of analytic functions,* Prentice-Hall Series in Modern Analysis, Prentice-Hall, Englewood Cliffs, N. J., 1962. MR **24** #A2844.

[H–W] K. Hoffman and J. Wermer, *A characterization of* $C(X)$, Pacific J. Math. **12** (1962), 941–944. MR **27** #325.

[K–S] H. König and G. Seever, *The abstract F. and M. Riesz theorem,* Duke Math. J. **36** (1969), 791–797.

[L 1] G. Lumer, *Algèbres de fonctions et espaces de Hardy,* Lecture Notes in Math., no. 75, Springer-Verlag, Berlin and New York, 1968. MR **39** #7434.

[L 2] ———, *Espaces de Hardy en pliesiers variable complexes,* C. R. Acad. Sci. Paris (to appear); *Hardy spaces in several complex variables* (in preparation).

[L 3] ———, *On Wermer's maximality theorem,* Invent. Math. **8** (1969), 236–237. MR **40** #4769.

[R] J. Rainwater, *A note on the preceding paper,* Duke Math. J. **36** (1969), 798–800.

[S–S] S. J. Sidney and E. L. Stout, *A note on interpolation,* Proc. Amer. Math. Soc. **19** (1968), 380–382. MR **36** #6944.

[V 1] N. Th. Varopoulos, *Ensembles pics et ensembles d'interpolation pour les algèbres uniformes,* C. R. Acad. Sci. Paris (to appear).

[V 2] ———, *Sur la reunion de deux ensembles d'interpolation d'une algèbre uniforme,* C. R. Acad. Sci. Paris (to appear).

[W] J. Wermer, *The space of real parts of a function algebra,* Pacific J. Math. **13** (1963), 1423–1426. MR **27** #6152.

[Z] L. Zalcman, *Analytic capacity and rational approximation,* Lecture Notes in Math., no. 50, Springer-Verlag, Berlin and New York, 1968. MR **37** #3018.

The following papers, while not specifically referred to (perhaps!) have appeared since the bibliography of [Gam] was formed. For earlier work the reader is referred to that bibliography.

P. Ahern, *A condition for peak points,* Duke Math. J. **37** (1970), 67–70. MR **41** #2025.

H. Alexander, *Polynomial approximation and analytic structure* (to appear).

C. Berger and W. Saffern, *Dirichlet enbalgebres of the disc algebra* (preprint).

J. -E. Björk, *Analytic structure in the maximal ideal space of a uniform algebra,* Ark. Mat. **8** (1970), 239–244;

A. Browder, *States, numerical range, etc.* Brown University, Providence, R. I. (preprint).

A. M. Davie, *Real annihilating measures for* $R(K)$, J. Functional Analysis **6** (1970), 357–386.

http://dx.doi.org/10.1090/cbms/011/05

A. M. Davie, *Bounded approximation and dirichlet sets,* J. Functional Analysis **6** (1970), 460–467.

———, *Dirichlet algebras of analytic functions,* J. Functional Analysis **6** (1970), 348–356. MR **42** #2300.

———, *Analytic capacity and approximation problems,* University of California, Los Angeles (preprint).

———, *Linear extension operators for spaces and algebras of functions,* University of California, Los Angeles (preprint).

A. M. Davie, T. W. Gamelin and J. Garnett, *Distance estimates and pointwise bounded density,* University of California, Los Angeles (preprint).

A. M. Davie and A. Stray, *Interpolation sets for analytic functions,* University of California, Los Angeles (preprint).

A. M. Davie and B. K. Øksendal, *Rational approximation on the union of sets,* University of California, Los Angeles (preprint); Proc. Amer. Math. Soc. (to appear).

———, *Peak interpolation sets for some algebras of analytic functions,* University of California, Los Angeles (preprint).

S. Fisher, *Norm-compact sets of representing measures,* Proc. Amer. Math. Soc. **19** (1968), 1063–1068. MR **38** #1531.

T. W. Gamelin, *Uniform algebras containing the polynomials* (to appear).

———, *Norm compactness of representing measures for* $R(K)$, J. Functional Analysis **3** (1969), 495–500. MR **39** #782.

———, *Localization of the corona problem* (to appear).

T. W. Gamelin and J. Garnett, *Constructive techniques in rational approximation,* Trans. Amer. Math. Soc. **143** (1969), 187–200. MR **40** #2882.

———, *Pointwise bounded approximation and hypodirichlet algebras,* Bull. Amer. Math. Soc. **77** (1971), 137–141. MR **42** #6491.

———, *Distinguished homeomorphisms and fiber algebras,* Amer. J. Math. **92** (1970), 455–474.

T. W. Gamelin and D. R. Wilken, *Closed partitions of maximal ideal spaces,* Illinois J. Math. **13** (1969), 789–795. MR **40** #4767.

J. Garnett, *Metric conditions for rational approximation,* Duke Math. J. **37** (1970), 689–699. MR **42** #4746.

———, *Interpolating sequences for bounded harmonic functions,* University of California, Los Angeles (preprint).

H. König, *Über das von Neumannsche Minimax-Theorem,* Arch. Math. **19** (1968), 482–487. MR **39** #1947.

A. Stray, *An approximation theorem for subalgebras of* H^∞, Pacific J. Math. **35** (1970), 511–515.

———, *Approximation and interpolation,* Pacific J. Math. (to appear).

———, *Interpolation by analytic functions* (preprint).

http://dx.doi.org/10.1090/cbms/011/05