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# AUTOMATA ON INFINITE OBJECTS AND CHURCH'S PROBLEM

by

**MICHAEL O. RABIN**

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(1) $$T(\mathfrak{A}_F) \cap V_{\{0,1\},\Lambda} = \{(\chi_A, T) \mid A \subseteq T,\ N_2 \models F(A)\};$$

$\chi_A$ is the characteristic function of $A$. Combined with Theorem 20 this immediately yields

THEOREM 23. *If $F(A)$ is a formula of $L_2$ such that $N_2 \models \exists \mathbf{A} \mathbf{F}(\mathbf{A})$, then there exists a regular set $A \subseteq T$ such that $N_2 \models F(A)$.*

PROOF. Let $T(\mathfrak{A}_F)$ satisfy (1), then $T(\mathfrak{A}_F) \neq \varnothing$. There exists, by Theorem 20, a regular tree $(v, T) \in T(\mathfrak{A}_F)$. The regular set $A = \{x \mid v(x) = 1\}$ satisfies $N_2 \models F(A)$.

A set $A \subseteq T$ is called *definable* in the second-order theory of $N_2$ if there exists a formula $F(A)$ of $L_2$ such that $N_2 \models \exists\,!\,AF(A)$. The corresponding notion of definability of a word $x \in T$ is trivial, because every $x \in T$ is definable in $N_2$.

COROLLARY 24. *A set $A \subseteq T$ is definable in $N_2$ if and only if $A$ is regular.*

PROOF. That every definable set is regular is a trivial consequence of the previous theorem.

Assume that $\Lambda \notin A \subseteq T$ is regular. For $x, y \in \{0, 1\}^\omega$, call $\langle x, y \rangle \in (\{0, 1\}^2)^\omega$ an *$A$-sequence* if $y(n) = 1$ if and only if $x(0) \cdots x(n) \in A$. The set of all $A$-sequences is definable by a $\{0, 1\}^2$ automaton $\mathfrak{B}$. By Lemma 15, the set of $T_\omega(\mathfrak{B})$-trees is automaton definable by a $\{0, 1\}$-automaton $\mathfrak{A}$.

By Theorem 1.10 of [8], there exists a formula $F(\mathbf{B})$ of $L_2$ such that $(\chi_B, T) \in T(\mathfrak{A})$ if and only if $N_2 \models F_{\mathfrak{A}}(B)$. Since $(\chi_B, T) \in T(\mathfrak{A})$ holds if and only if $B = A$, it follows that $F_{\mathfrak{A}}(\mathbf{B})$ defines $A$.

Let $M = \langle A, R_\lambda \rangle_{\lambda < \alpha}$ be a structure, $L$ the appropriate *first-order* language. An element $a \in A$ is *definable* in $M$ if for some formula $F(\mathbf{x})$ of $L$, $M \models \exists\,! xF(x)$ and $M \models F(a)$ hold. The following statement and its proof are closely related to, but not quite identical with, the first part of Theorem 2 of [6]. The proof is a straightforward application of the Tarski-Vaught characterization of elementary extensions and wili be omitted.

LEMMA 25. *Denote by $D$ the set of all definable elements of $M$. If for every formula $F(\mathbf{x})$ satisfying $M \models \exists \mathbf{x} F(\mathbf{x})$ there exists an $a \in D$ such that $M \models F(a)$, then $M \restriction D = \langle D, R_\lambda \restriction D \rangle_{\lambda < \alpha} \prec M$. If $D \neq \varnothing$ then the converse also holds (trivial).*

By the usual device of passing from $N_2$ to a structure $M$ with domain $T \cup P(T)$ and adding $\in$ to the relations, we may consider the monadic second-order theory of $N_2$ as the first-order theory of $M$. Let

$$R = \{A \mid A \subseteq T,\ A \ \text{regular}\}.$$

Then the definable elements of $M$ are precisely the words $x \in T$ and the sets $A \in R$. Combined with Theorem 23, Corollary 24, and Lemma 25, this yields the following "basis" result.

THEOREM 26. *If $F(\mathbf{A}_1, \cdots, \mathbf{A}_n, \mathbf{x}_1, \cdots, \mathbf{x}_m)$ is a formula of $L_2$ and if $A_1, \cdots, A_n \in R, x_1, \cdots, x_n \in T$, then $F(\mathbf{A}_1, \cdots, \mathbf{A}_n, \mathbf{x}_1, \cdots, \mathbf{x}_m)$ is true in $N_2$ if and only*

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*if it is true when all set-quantifiers are relativized to* $R$. *In particular a sentence* $F$ *is true in* $N_2$ ($F \in S2S$) *if and only if it is true with all set-quantifiers relativized to* $R$.

These results have many interesting applications to second-order theories. For example

THEOREM 27. *Let* $F$ *be a sentence of the (monadic) second-order language of linearly ordered sets, which is not true in all countable linearly ordered sets, then there exists a regular set* $A \subseteq \{0, 1\}^*$ *so that* $\langle A, \preceq \rangle \models \sim F$.

PROOF. In [8, p. 11] we have shown that for every countable linearly ordered set $\langle B', \leqslant \rangle$ there exists a set $B \subseteq \{0, 1\}^*$ satisfying $\langle B', \leqslant \rangle \approx \langle B, \preceq \rangle$. Let $F(\mathbf{A})$ be the formula of $L_2$ obtained from $F$ by replacing all occurrences of $\leqslant$ by $\preceq$, relativizing all individual quantifiers to $A$ and all set-quantifiers to subsets of $A$. Then $N_2 \models \exists \mathbf{A} \sim F(\mathbf{A})$. Hence there exists an $A \in R$ such that $N_2 \models \sim F(A)$. This implies $\langle A, \preceq \rangle \models \sim F$.

Similar applications of Theorems 23 and 26 will yield basis theorems and the existence of very constructive counterexamples (defined by regular sets) for non-theorems, for all the theories proved decidable in [8]. This includes the monadic second-order theory of a unary function; the restricted monadic second-order theory of countable boolean algebras (with the set variables restricted to ideals), and others. In particular, this also produces an interesting denumerable non-standard model of the first-order theory of the lattice of all closed subsets of the real line. These results will be described in detail in a subsequent publication.

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