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Banach Algebra Techniques in the Theory of Toeplitz Operators

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PREFACE

During the week of June 12–16, 1972, I gave a series of ten lectures at the University of Georgia on the occasion of a Regional Conference sponsored by the Conference Board of the Mathematical Sciences with the support of the National Science Foundation. These notes are a corrected version of the lecture notes which were distributed at that time.

The theme of the lectures was the use of techniques drawn from the theory of Banach algebras to study Toeplitz operators. An attempt was made at unifying diverse results, and point of view and direction were stressed rather than completeness. In particular, many recent results and problems were discussed.

I would like to thank Bernard Morrel who planned and arranged the conference and the University of Georgia which provided the facilities.

Stony Brook
July, 1972

R. G. Douglas

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$$T_\phi T_\phi^* = \begin{pmatrix} T_{z^n}T_{z^n}^* + T_{w^m}T_{w^m}^* & 0 \\ 0 & 2I \end{pmatrix} \quad\text{and}\quad T_\phi^* T_\phi = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Therefore T_ϕ is seen to be Fredholm, $\dim\ker T_\phi = \dim\ker T_\phi^* T_\phi = 0$, and $\dim\ker T_\phi^* = \dim\ker T_\phi T_\phi^* = \dim\ker (T_{z^n}T_{z^n}^* + T_{w^m}T_{w^m}^*) = mn$. Thus we obtain that $\operatorname{ind}_z(T_\phi) = -mn$, and we see that even Toeplitz operators with matrix coefficients can have nonzero index.

These examples show that computing the index for Fredholm operators in $\mathcal{T}(C_{M_m}(\mathbf{T}^2))$ is not a trivial problem. We consider first the case of operators in $\mathcal{T}(C(\mathbf{T}^2))$. If T is an operator in $\mathcal{T}(C(\mathbf{T}^2))$ which is Fredholm, then $\gamma_z(T) \oplus \gamma_w(T)$ is an invertible element of $C(\mathbf{T}, \mathcal{T}(C(\mathbf{T}))) \oplus C(\mathbf{T}, \mathcal{T}(C(\mathbf{T})))$. Moreover, we have from Theorem 17 that $\{\pi[\gamma_z(T)(z)]\}(w) = \{\pi[\gamma_w(T)(w)]\}(z)$. Thus the symbol space for this algebra is the collection Σ_1 of invertible elements of $C(\mathbf{T}, \mathcal{T}(C(\mathbf{T}))) \oplus C(\mathbf{T}, \mathcal{T}(C(\mathbf{T})))$ satisfying this identity. To compute the index one can concentrate on the component structure of Σ_1. This is easier to determine for the scalar case. If ϕ is the function in $C(\mathbf{T}^2)$ defined by $\phi(z, w) = \{\pi[\gamma_z(T)(z)]\}(w) = \{\pi[\gamma_w(T)(w)]\}(z)$, then $\phi \neq 0$ and is homotopic to a constant. Let ϕ_t be a nonzero function in $C(\mathbf{T}^2)$, for $0 \leqslant t \leqslant 1$, continuous in t such that $\phi_0 \equiv 1$ and $\phi_1 \equiv \phi$. Each of the operators $T_{1/(\phi_t)}$ is Fredholm on $H^2(\mathbf{T})$ and the operator S_t in $\mathcal{T}(C(\mathbf{T}^2))$ defined by $S_t = T_{1/(\phi_t)}T$ is Fredholm, and hence S_1 has the same index as T.

The important property of the symbol $\gamma_z(S_1) \oplus \gamma_w(S_1)$ is that $\pi[\gamma_z(S_1)(z)](w) = \pi[\gamma_w(S_1)(w)](z) = 1$ for (z, w) in $\mathbf{T}^2$. Therefore, $\gamma_z(S_1) - I$ and $\gamma_w(S_1) - I$ are compact operators on $H^2(\mathbf{T})$ for (z, w) in $\mathbf{T}^2$. If $\mathcal{K}$ denotes the group of invertible operators on $H^2(\mathbf{T})$ having the form $I + K$ for some K in $\mathcal{LC}(H^2(\mathbf{T}))$, then F_1 and G_1 lie in $[\mathbf{T}, \mathcal{K}]$. Since $[\mathbf{T}, \mathcal{K}]$ is isomorphic to $\mathbf{Z}$, essentially using the determinant function, there are integers $\operatorname{ind}_t(\gamma_z(S_1))$ and $\operatorname{ind}_t(\gamma_w(S_1))$ naturally associated with the symbol for S_1. Thus what we have is that the group of homotopy classes of the symbol space Σ_1 is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ and the problem is reduced to determining a homomorphism from $\mathbf{Z} \oplus \mathbf{Z}$ to $\mathbf{Z}$.

It is possible to compute the index of operators with symbols $\gamma_z(S_1) \oplus 1$ and $1 \oplus \gamma_w(S_1)$ separately since the symbols lie in Σ_1 and then to take the product. Let us recall the operator $R = T_z T_z^* + \tfrac{1}{2} T_w^*$ considered earlier. Computing we have $\gamma_z(R)(z) = I + \tfrac{1}{2} T_w^*$ and $\gamma_w(R)(w) = T_z T_z^* + \bar{w}/2$ and

$$\pi\{[\gamma_z(R)(z)]\}(w) = 1 + \bar{w}/2 = \tfrac{1}{2}(2 + \bar{w}).$$

If we multiply by $T_{2/(2+\bar{w})}$, then we obtain $\gamma_z(T_{2/(2+\bar{w})}R) = 1$ and $\gamma_w(T_{2/(2+\bar{w})}R)(w) = I + E_0/(1 + 2w)$, where E_0 is the projection onto the constant functions on $H^2(\mathbf{T})$. To compute $\operatorname{ind}_t\{\gamma_w(T_{2/(2+\bar{w})}R)\}$ it is sufficient to take the winding number of the function $1/(1 + 2w)$ on $\mathbf{T}$ which is -1. Since the index of R was shown to be 1 we obtain that

$$\operatorname{ind}_a(R) = -(\operatorname{ind}_t[\gamma_z(T_{2/(2+\bar{w})}R)] + \operatorname{ind}_t[\gamma_w(T_{2/(2+\bar{w})}R)]).$$

But this computation is enough to complete the proof.

If Σ_1° denotes the subgroup of Σ_1 consisting of symbols $F \oplus G$ for which the range of both F and G lies in K, then we have shown that the homotopy classes of Σ_1° and Σ_1 coincide. Thus there exists an isomorphism ind_t from the homotopy classes of Σ_1 to $\mathbf{Z} \oplus \mathbf{Z}$. Let σ denote the homomorphism from $\mathbf{Z} \oplus \mathbf{Z}$ onto $\mathbf{Z}$ defined by $\sigma(m, n) = m + n$.

We can now state the index result due to Coburn, Singer and the author [16] for the scalar case.

THEOREM 19. *If T is an operator in $\mathcal{T}(C(\mathbf{T}^2))$, then T is Fredholm if and only if $\gamma_z(T) \oplus \gamma_w(T)$ lies in Σ_1 in which case*

$$\text{ind}_a(T) = -\sigma\{\text{ind}_t[\gamma_z(T) \oplus \gamma_w(T)]\}.$$

The extension of this result to the matrix case involves considerably more sophisticated topological arguments. We state the result and refer the reader to [16] for details.

PROPOSITION 10.2. *If T is a Fredholm operator in $\mathcal{T}(C_{M_k}(\mathbf{T}^2))$ with symbol $\gamma_z(T) \oplus \gamma_w(T)$ in Σ_k, then there is a path $F_t \oplus G_t$ in Σ_k such that $F_0 = \gamma_z(T)$, $G_0 = \gamma_w(T)$ and such that*

$$F_1(z) = \begin{pmatrix} z^m & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \quad \text{and} \quad G_1(w) = \begin{pmatrix} w^n & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

for some (m, n) in $\mathbf{Z}^2$. The index of T is given by $\text{ind}_a(T) = -(m + n)$.

In general the path is not unique nor are m and n uniquely determined for $k > 1$.

Lastly, we consider the following invertibility result [25] for Toeplitz operators on the bidisk.* It generalizes a recent result of Malyšev [49] by removing the requirement that the function have an absolutely convergent Fourier series and allowing more general functions. We refer the reader to [25] for more details.

THEOREM 20. *If ϕ is a function continuous on the closed bidisk $\mathbf{D}^2$ which is holomorphic on the interior, (a_1, a_2) is a point in $\mathbf{D}^2$, and ψ is defined on $\mathbf{T}^2$ by*

$$\psi(z_1, z_2) = \frac{\phi(z_1, z_2)}{(z_1 - a_1)(z_2 - a_2)};$$

then T_ψ is invertible if and only if $\psi \neq 0$ and is homotopic to a constant.

PROOF. We begin by showing how to reduce to the case where $(a_1, a_2) = (0, 0)$. There are several ways to do this. The most straightforward is to observe that if η is the conformal self map of the bidisk defined by

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* In the original version of these notes this theorem was stated for Toeplitz operators on the n-dimensional polydisk. The proof, however, had a gap. It is unknown whether or not the theorem is valid in that generality.

$$\eta(z_1, z_2) = \left(\frac{z_1 - b_1}{1 - \bar{b}_1 z_1}, \frac{z_2 - b_2}{1 - \bar{b}_2 z_2} \right)$$

for a fixed (b_1, b_2) in $\mathbf{D}^2$, then T_θ is unitarily equivalent to $T_{\theta \circ \eta}$ for every continuous function θ on $\mathbf{T}^2$. To prove this it is sufficient, in view of the tensor product representation in [26], to consider the one-dimensional case; and for this it suffices to apply Coburn's uniqueness result [12] to the isometry $T_{(z-b)/(1-\bar{b}z)}$ on $H^2(\mathbf{T})$.

If T_ψ is invertible on $H^2(\mathbf{T}^2)$, then it follows from [26] that ψ is nonvanishing on $\mathbf{T}^2$ and homotopic to a constant. The converse involves showing that such an operator has no kernel.

Suppose f is a function in $H^2(\mathbf{T}^2)$ lying in the kernel of T_ψ. Then we have $T_\psi f = T^*_{z_1 z_2}(\phi f) = 0$, and hence there exist functions g_1, g_2 in $H^2(\mathbf{T})$ such that

$$\phi(z_1, z_2) f(z_1, z_2) = g_1(z_1) + g_2(z_2).$$

We want to show that this is impossible. First, there exists $\{z : 1 > |z| > 1 - \epsilon\}$, such that $\phi(z_1, z_2) \neq 0$ for (z_1, z_2) in $\mathbf{A}^2$. This follows from the continuity of ϕ on $\bar{\mathbf{D}}^2$ and the fact that $\phi(z_1, z_2) \neq 0$ for (z_1, z_2) in $\mathbf{T}^2$. Now fix z_2 in $\mathbf{A}$ and consider the holomorphic function defined $\phi_1(z_1) = \phi(z_1, z_2)$ for z_1 in $\mathbf{D}$. Since the index of the curve $\phi_1(z)$ traced by z in $\mathbf{T}$ is 1, there exists a unique complex number $\xi(z_2)$ of modulus less than $1 - \epsilon$ such that $\phi(\xi(z_2), z_2) = 0$. This follows from the fact that $\phi(z_1, z_2)/z_1 z_2$ is homotopic to a constant on $\mathbf{T}^2$. Moreover ξ is a holomorphic function from $\mathbf{A}$ to the open disk $\mathbf{D}_{1-\epsilon}$ of radius $1 - \epsilon$.

Since a function in $H^2(\mathbf{T}^2)$ can have no poles in $\mathbf{D}^2$ we see that $\phi(z_1, z_2) = 0$ for (z_1, z_2) in $\mathbf{D}^2$ implies $g_1(z_1) + g_2(z_2) = 0$, and hence we have the inclusion

$$g_2(A) \subset -g_1(\mathbf{D}_{1-\epsilon}).$$

For X an arbitrary subset of C let $X^\#$ denote the subset of $\mathbf{C}$ obtained by taking the closure of X and adding the bounded open components in the complement. It is easy to verify that $g(\mathbf{A})^\# \supset g(\mathbf{D})$ for g holomorphic on $\mathbf{D}$.

Thus we obtain

$$[-g_1(\mathbf{D}_{1-\epsilon})]^\# \supset g_2(\mathbf{D}).$$

Therefore the maximum of the function g_1 on $\mathbf{D}_{1-\epsilon}$ is at least as great as that of g_2 on $\mathbf{D}$. Hence the maximum of g_1 on $\mathbf{D}$ exceeds that of g_2 on $\mathbf{D}$ unless g_1 is constant. But this implies a contradiction. Otherwise we repeat the preceding argument with the roles of z_1 and z_2 reversed to obtain a contradiction. In either case the theorem is proved.

We conclude with a comment. In [52] Pattanayak shows that the collection of functions ϕ in $C_{M_m}(\mathbf{T}^2)$ for which T_ϕ is Fredholm is a dense open set in the collection of invertible functions which are homotopic to a constant. Thus, for $C_{M_m}(\mathbf{T}^2)$ the generic case is Fredholm. On the other hand, this is not true for $C_{M_2}(\mathbf{T}^3)$.

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