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Number 16

# Measure Algebras

Joseph L. Taylor

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## Preface

These notes were prepared in conjunction with the N. S. F. regional conference on measure algebras held at the University of Montana during the week of June 19, 1972.

Our original objective in preparing these notes was to give a coherent detailed and simplified presentation of a body of material on measure algebras developed in a recent series of papers by the author (Taylor [1] – [10]). This material has two main thrusts: the first concerns an abstract characterization of Banach algebras which arise as algebras of measures under convolution (convolution measure algebras) and a semigroup representation of the spectrum (maximal ideal space) of such an algebra; the second deals with a characterization of the cohomology of the spectrum of a measure algebra and applications of this characterization to the study of idempotents, logarithms, and invertible elements.

As this project progressed the original concept broadened. The final product is a more general treatment of measure algebras, although it is still heavily slanted in the direction of our own work.

Chapter 1 contains a brief introductory discussion of convolution and the structure of the algebras $L^1(G)$ and $M(G)$, as well as an introduction to several of the problems which will be solved or partially solved in later chapters.

Chapters 2 and 3 are devoted to a development and discussion of convolution measure algebras and to a representation theorem for the spectrum of such an algebra. Several examples of convolution measure algebras are discussed in Chapter 4. Much of the material of Chapters 2–4 is contained in Taylor [1] and can be skipped by readers familiar with that paper. However, our discussion here is considerably more detailed and does not assume familiarity with Kakutani's $L$-space theory or the theory of topological semigroups.

Chapters 5–9 are mainly concerned with a characterization of the cohomology of the spectrum of a measure algebra and applications to the study of idempotents, logarithms, and inverses in such an algebra. This material originally appeared in Taylor [3] – [10]. The development here has been considerably simplified.

Chapter 10 is largely independent of Chapters 4–9. In it we discuss some results of Miller [1] on Gleason parts in a measure algebra, of Taylor [2] and Johnson [3] on the Shilov boundary of $M(G)$, and of Brown and Moran [3] on infinite product measures.

We would like to express our gratitude to Michael J. Fisher who conceived and organized the conference, and to the National Science Foundation, the University of Montana, the Rocky Mountain Mathematics Consortium, and the Conference Board of the Mathematical Sciences. We also acknowledge the generous financial support of the Air Force Office of Scientific Research under grant No. 1313-67, the National Science Foundation under grant No. GP-32331, and the Alfred P. Sloan Foundation.

University of Utah

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