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Preface

This paper organizes and expounds the lectures which the author delivered at the Regional Conference held 7–11 June 1971 at the University of New Hampshire under the sponsorship of the Conference Board of the Mathematical Sciences and with the support of the National Science Foundation.

The subject is taken from the theory of unitary dilations of contraction operators and of the theory of functional models of such operators. The fundamental result in this area, namely the existence and uniqueness of the minimal unitary dilation for a contraction, was proved by the author in 1953 and was the starting point of various far-reaching investigations of the structure and properties of Hilbert space operators. This trend is far from being exhausted. Most of the investigations in this direction were made by the author in close and lasting collaboration with Ciprian Foiaş; our book on *Harmonic analysis of operators on Hilbert space,** quoted in the sequel by [H], gives a detailed account of our results as well as of a good part of the relevant results of a number of other mathematicians.

In my lectures I selected some of the chapters of the theory which I expounded in some detail, other chapters were only sketched, and many others not even mentioned. I have tried to be quite detailed in the geometric (or purely operator.theoretic) aspects of the theory, thus preparing a natural and rapid introduction, by "Fourier representations," of the characteristic functions and functional models of operators. I have tried to make clear the connections of the invariant subspace problem with the factorization problem of the characteristic function, but I could not enter into such applications as the fairly complete spectral theory of weak contractions. I have included the elements of the functional calculus of contractions and of the theory of operators of C_0, and have indicated the main results in the Jordan model theory of these operators, but I have not included one-parameter continuous semigroups, or the way of passing from contractions to (not necessarily bounded) dissipative or accretive operators. I have given some attention to the "lifting theorem," which is a most useful tool in many applications, but I have not even mentioned applications of the theory to scattering

* North-Holland and Akadémiai Kiadó, 1970. Revised and enlarged edition of the first edition, in French, published by Masson et Akadémiai Kiadó in 1967. A Russian edition of the revised variant was published in 1970 by "Mir" in Moscow.

theory or to prediction theory, etc. I have indicated the problem and the most interesting results concerning unitary dilations of a commutative system of contractions but have excluded any reference to dilations of representations of function algebras, or to unitary ρ-dilations.

Thus, in short, I did not endeavour to give a bird's eye view of *all* the realm of operator theory connected with the dilation concept; instead I tried to give a possibly clear picture of some of its main provinces. I hope that some of the readers will find the theme interesting and promising enough for further study, with the help of the book [H] and the current literature.

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July 1973

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if $S(\Theta)$ and $S(ab)$ are similar. Then, applying the "multiplication property" of the lifting (§7.3) we have, for any $w \in \mathfrak{H}(ab)$,

$$w = X'Xw = P_{\mathfrak{H}(ab)}[y'_1, y'_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} w$$

$$= (y'_1 y_1 + y'_2 y_2)w + (\text{an element of } abH^2).$$

Choosing, in particular,

$$w(\lambda) = 1 - \overline{a(0)}\,\overline{b(0)}\,a(\lambda)\,b(\lambda)$$

(cf. (9.2.2)) and using (10.1.5) we infer that $1 = a\xi + b\eta + ab\zeta$ with some $\xi, \eta \in H^\infty$ and $\zeta \in H^2$. As $\zeta = \bar{a}\bar{b} - \bar{b}\xi - \bar{a}\eta$ on the unit circle we also have $\zeta \in H^\infty$. Hence,

$$1 = ax + by \tag{10.1.6}$$

with some $x, y \in H^\infty$. Thus the existence of $x, y \in H^\infty$ satisfying the equation (10.1.6) is a *necessary* condition for $S(a) \oplus S(b)$ to be *similar* to $S(ab)$.

A case when this condition is certainly *not* fulfilled is the following:

a: a Blaschke product with real zeros λ_n converging to 1, say with $\lambda_n = 1 - 1/n^2$,
b: the "singular" inner function $\exp[(\lambda + 1)/(\lambda - 1)]$.

Indeed, in this case a and b tend to 0 on the points λ_n (as $n \to \infty$) and so does $ax + by$ for every $x, y \in H^\infty$, contradicting (10.1.6).

Thus in this case the operators $S(a) \oplus S(b)$ and $S(ab)$ are *not similar.*

But it turns out that these operators are *quasi-similar* for every relatively prime a, b. Indeed by choosing $y_1 = y_2 = 1$ and $y'_1 = b$, $y'_2 = a$ (see (10.1.5)), the operators X and X' are quasi-affinities. To prove this we argue as follows. First recall the definitions:

$$Xw = P_{\mathfrak{H}(\Theta)}\begin{bmatrix} w \\ w \end{bmatrix} = \begin{bmatrix} P_{\mathfrak{H}(a)}w \\ P_{\mathfrak{H}(b)}w \end{bmatrix} \quad \text{for } w \in \mathfrak{H}(ab),$$

$$X'\begin{bmatrix} u \\ v \end{bmatrix} = P_{\mathfrak{H}(ab)}(bu + av) \qquad \text{for } \begin{bmatrix} u \\ v \end{bmatrix} \in \mathfrak{H}(\Theta), \text{ i.e., } u \in \mathfrak{H}(a),\ v \in \mathfrak{H}(b).$$

(1) $Xw = 0$ implies $w \in aH^2 \cap bH^2 = abH^2$; as $w \in \mathfrak{H}(ab)$ this is possible only if $w = 0$. Thus X has zero kernel.

(2) If $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathfrak{H}(\Theta)$ is orthogonal to Xw for every $w \in \mathfrak{H}(ab)$, then $(u + v, w) = (u, P_{\mathfrak{H}(a)}w) + (v, P_{\mathfrak{H}(b)}w) = \left(\begin{bmatrix} u \\ v \end{bmatrix}, Xw\right) = 0$ for every $w \in \mathfrak{H}(ab)$, and hence $u + v \in abH^2$. On the other hand, $u \perp aH^2$ and $v \perp bH^2$ imply $u + v \perp abH^2$. Hence, $u + v = 0$. Therefore, both u and v are orthogonal to aH^2 and bH^2, and hence to $aH^2 \vee bH^2$, i.e., to H^2. Thus, $u = v = 0$: the range of X is dense in $\mathfrak{H}(\Theta)$.

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(3) Suppose $X'\begin{bmatrix} u \\ v \end{bmatrix} = 0$ for a $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathfrak{H}(\Theta)$; that is, suppose $bu + av \in abH^2$. As on the other hand $u \in \mathfrak{H}(a)$, $v \in \mathfrak{H}(b)$ imply $u \perp aH^2$, $v \perp bH^2$ and therefore $bu \perp abH^2$ and $av \perp abH^2$, we have $bu + av \perp abH^2$. Thus $bu + av = 0$. As a, b are prime we infer that a is a divisor of u and b is a divisor of v, and hence $u \in aH^2$, $v \in bH^2$. We conclude that $u = v = 0$: X' has zero kernel.

(4) Let $w \in \mathfrak{H}(ab)$ be orthogonal to the range of X', i.e., orthogonal to $bu + av$ for every $u \in \mathfrak{H}(a)$ and $v \in \mathfrak{H}(b)$. Then $w \perp b\mathfrak{H}(a)$; and as $w \perp baH^2$ we also have $w \perp b(\mathfrak{H}(a) \oplus aH^2)$, i.e., $w \perp bH^2$. Analogously, $w \perp aH^2$. As $aH^2 \vee bH^2 = H^2$ because a, b are prime, we get $w = 0$. Hence X' has dense range.

Summarizing, we have proved: *If a, b are any two prime inner functions then $S(a) \oplus S(b)$ is quasi-similar to $S(ab)$, but if the equation $1 = ax + by$ has no solution $x, y \in H^\infty$ then $S(a) \oplus S(b)$ is not similar to $S(ab)$.*

This example shows that with operators on infinite-dimensional Hilbert spaces quasi-similarity is a weaker, but apparently more natural, relation than similarity. In particular, Theorem A of §9 does not hold with similarity in place of quasi-similarity.

2. In Theorems B and C of §9 we were concerned with operators A which admit a representation $A = \phi(T)$, with the given operator $T \in C_0$ and with some function ϕ of class N_T. Since the operators A are bounded, it is natural to ask whether they also admit a representation $A = w(T)$ with some function $w \in H^\infty$. That this is in general *not* the case will be shown by the following example, of considerable interest in itself.

Let again a, b be two, relatively prime inner functions and set $T = S(a) \oplus S(b)$. Then $m_T = ab$. Note that $a + b$ and ab have no nonconstant inner divisors, and hence by Corollary 3 in §8.3 the operator $(a + b)(T)$ is a quasi-affinity. Thus the function $\phi = (a - b)/(a + b)$ belongs to the class N_T. The corresponding operator $V = \phi(T)$ turns out to be bounded, indeed a symmetry; namely $V = (-I_{\mathfrak{H}(a)}) \oplus I_{\mathfrak{H}(b)}$. For

$$
\begin{aligned}
(a - b)(T) \cdot V - (a + b)(T) \\
&= [-(a - b) - (a + b)](S(a)) \oplus [(a - b) - (a + b)](S(b)) \\
&= -2[a(S(a)) \oplus b(S(b))] = 0 \oplus 0 = 0.
\end{aligned}
$$

But, in general, V cannot be represented in the form $V = w(T)$, with $w \in H^\infty$. For if it *can*, then we have $-I_{\mathfrak{H}(a)} = w(S(a))$ and $I_{\mathfrak{H}(b)} = w(S(b))$; considering in particular the functions $1 - \overline{a(0)}a \in \mathfrak{H}(a)$ and $1 - \overline{b(0)}b \in \mathfrak{H}(b)$ we infer that

$$
-1 + \overline{a(0)}a - w \cdot (1 - \overline{a(0)}a) \in aH^2,
$$

$$
1 - \overline{b(0)}b - w \cdot (1 - \overline{b(0)}b) \in bH^2,
$$

and hence

$$
1 + w = au, \qquad 1 - w = bv,
$$

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where $u, v \in H^2$; indeed $u, v \in H^\infty$. Thus we obtain that a *necessary condition for* V *to have a representation* $V = w(T)$ *with some* $w \in H^\infty$ *is that the equation* $1 = ax + by$ *admit a solution* $x, y \in H^\infty$.

As we know, there are relatively prime a, b, for which this equation has no such solution.

This example indicates that our functional calculus with cnu contractions T, constructed in §8 for the class N_T, is natural enough; indeed the class of bounded analytic functions would not suffice.

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The general reference is the monograph [H] quoted in the Preface, where detailed references and historical comments can also be found.

Section 1. Chapter I in [H]. The first paper on the (strong) unitary dilation is [9], where a completely different proof was given. The construction given in the text is essentially that of [8].

Section 2. Chapter II in [H]. See also [5].

Section 3. Chapter VI in [H].

Section 4. Chapter VI and Chapter IX.1 in [H].

Section 5. Chapter VII in [H]. For "strange" factorizations, see [3].

Section 6. Chapter I in [H]. Further references: [1], [2], [6], [15].

Section 7. Chapters II.2 and VI.8 in [H]. Investigations in this direction were started by the paper [7]. There are many applications of these theorems; see e.g. [14].

Section 8. Chapters III and IV in [H].

Section 9. Chapters III, VIII, and IX in [H] and [10]–[14].

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