

Conference Board of the Mathematical Sciences

CBMS

Regional Conference Series in Mathematics

Number 21

Holomorphic Functions
of Finite Order
in Several Complex Variables

Wilhelm Stoll



American Mathematical Society
with support from the
National Science Foundation



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Dedicated to the founders of this field in several variables.

Stefan Bergman Hellmuth Kneser Pierre Lelong

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Preface

From June 18 to 22, 1973 a National Science Foundation Regional Conference on holomorphic functions of finite order in several variables was held at the University of Wisconsin-Whitewater, Wisconsin. In one variable, the theory of functions of finite order is well developed and a number of expositions exist. In several variables, the theory has not reached this stage of maturity but the basic principles and ideas are emerging.

The allotment of ten hours of lecture during the conference week did not permit to survey in depth all the principal features of the theory. Therefore one fundamental aspect was selected as the central topic of this survey. As such I selected the construction of holomorphic functions with growth estimates to given zero sets. This topic is relatively easily accessible and has many similarities to the one variable theory. A coherent account is lacking. Important new results have been obtained and the recall of older results is of value. Other aspects are better covered. For instance, during the spring of 1973 a special semester in value distribution in several variables was held at Tulane University. The proceedings will provide a wide panorama of the investigations into the properties of a holomorphic map. In view of these reasons this monograph will concentrate on the construction of holomorphic functions with growth estimates to given zero sets. As an exception to this rule, the two main theorems of value distribution will be stated here for a meromorphic map of a hermitian vector space into a complex projective space, because of the fundamental importance of these theorems. After three preparatory sections, this will be done in §4.

In §5, the Jensen-Poisson formula for balls will be proved and applied to the construction of a canonical function on a relatively compact ball in the domain of the given divisor. In §6, the canonical function for a divisor in a hermitian vector space will be constructed using weights. The connection to the canonical function of Lelong [28] is established; however, I will not pursue the theory of Lelong and his school further. For a survey see Lelong [29], [30]. The results on the canonical function are applied to construct theta functions to periodic divisors. Rubel and Taylor [36] had introduced a Fourier series method to study and construct functions of finite λ -type in one variable. Kujala [23] extended this investigation to several variables. A short survey of his results is given in §8. Time limitations did exclude this topic from being presented at the conference. In §9, the canonical function of a divisor of finite order on the unit ball is introduced and a factorization theorem is given. §9 reviews results contained in the dissertation of Mueller [31]. Again, consider the case of a nonnegative divisor ν on a hermitian vector space. Ronkin [35] gives a third integral representation of the canonical function of ν . In §10, Ronkin's results will be completely

restructured and considerably advanced. Ronkin uses a power series method similar to Kneser's. Here a different method will be used. A Jensen-Poisson formula for polydiscs will be established and analogues to the Weierstrass product and the Blaschke product will be given. An integral representation of a good inner function on a polydisc will be obtained. Since a good part of §10 is new, complete proofs will be given here. In §11, a survey of fundamental results of Skoda [44] will be given. Skoda uses ingenious new ideas, extensive calculations and deep results of Hörmander.

Hopefully, this survey will make the topic better known and easier accessible and will, in consequence, stimulate new research into the many open questions remaining.

I would like to express my gratitude to Professor Rudolph Najar for his kind invitation and his devoted efforts in organizing this conference.

WILHELM STOLL
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