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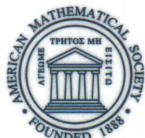
CBMS

Regional Conference Series in Mathematics

Number 23

Lectures on Set Theoretic Topology

Mary Ellen Rudin



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LECTURES ON SET THEORETIC TOPOLOGY

by
MARY ELLEN RUDIN

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