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Class Groups and Picard Groups of Group Rings and Orders

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CONTENTS

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Now let C denote the center of Λ, and consider those invertible (Λ, Λ)-bimodules M such that

$$cm = mc \quad \text{for all } m \in M,\ c \in C. \tag{9.6}$$

The classes (M) of such bimodules form a subgroup Picent Λ of Pic Λ. If Γ is Morita equivalent to Λ, then Picent $\Gamma \cong$ Picent Λ (see MO, (37.9)). We note also (MO, (37.21) and (37.22)):

(9.7) THEOREM. (i) Picent $A = 1$ *for any semisimple artinian ring* A.

(ii) Picent $\Lambda = 1$ *for any commutative semilocal*[11] *ring* Λ.

From now on, let Λ be an R-order in a semisimple F-algebra A, where char $F = 0$, and let C denote the center of Λ. We wish to make the bimodule isomorphism classes of two-sided Λ-ideals in A into a multiplicative group, but this cannot be done because of the absence of inverses. We must therefore restrict the types of ideals considered: an *invertible* Λ-ideal is a two-sided Λ-module $M \subset A$, such that $MN = NM = \Lambda$ for some two-sided Λ-module $N \subset A$. It follows at once that there are bimodule surjections

$$M \otimes_\Lambda N \longrightarrow \Lambda, \qquad N \otimes_\Lambda M \longrightarrow \Lambda,$$

and therefore (MO, (16.7)) the bimodule ${}_\Lambda M_\Lambda$ is invertible, with inverse N. We may remark that

$$N = \operatorname{Hom}_\Lambda({}_\Lambda M, {}_\Lambda \Lambda) \cong \{a \in A : Ma \subset \Lambda\}.$$

Each invertible Λ-ideal M obviously satisfies condition (9.6). Conversely, it turns out that every invertible (Λ, Λ)-bimodule, for which (9.6) holds, is bimodule isomorphic to an invertible Λ-ideal in A. Thus Picent Λ may be described as the group of classes of invertible Λ-ideals in A, and as such, is a natural generalization of the ideal class group Cl R. In fact, we obtain easily:

$$\text{Picent } \Lambda \cong I(\Lambda)/\{\Lambda c : c \in u(FC)\}, \tag{9.8}$$

where $I(\Lambda)$ is the group of invertible Λ-ideals in A, and FC is the center of A.

(The fact that the elements of Picent Λ are represented by Λ-ideals is a consequence of (9.7(i)). Condition (9.6) is needed in order to permit the use of the Skolem-Noether Theorem: Every automorphism of A which is the identity map on FC must be an inner automorphism.)

Every two-sided Λ-ideal in A is of course also a left Λ-ideal, and bimodule isomorphism implies isomorphism. Hence if F is an algebraic number field, it follows from the Jordan-Zassenhaus Theorem (1.7) that Picent Λ is finite.

In certain cases, we can compute Picent Λ directly from (9.8). For example, we have (MO, (37.27)).

(9.9) THEOREM. *Let A be a simple algebra whose center F is an algebraic number field, and let Λ be a maximal R-order in A. For a prime ideal P of R, let $A_P \cong M_n(D)$, where D is a skewfield, and set $m_P = (D : F_P)^{1/2}$, the* local index *of A at P. Then* Picent Λ_P

[11] See (2.2).

http://dx.doi.org/10.1090/cbms/026/09

is a cyclic group of order m_P. *Furthermore,* $m_P = 1$ *for almost all* P.

OUTLINE OF PROOF. The group $I(\Lambda_P)$ is an infinite cyclic group generated by rad Λ_P. If π denotes a prime element of R_P, then $\pi\Lambda_P = (\text{rad } \Lambda_P)^{m_P}$. Since $FC = F_P$ in this case, the denominator on the right-hand side of (9.8) consists of all powers $\pi^r\Lambda_P$, $r \in \mathbf{Z}$. Therefore Picent $\Lambda_P \cong \mathbf{Z}/m_P\mathbf{Z}$, as asserted. The fact that $m_P = 1$ a.e. follows from an easy argument involving discriminants (see MO, (25.7)).

The calculation of Picent Λ_P, in case Λ_P is not a maximal order, is rather difficult. One approach involves the calculation of the automorphisms of Λ_P, as we now explain. Let Aut Λ denote the group of automorphisms of Λ, and In Λ the subgroup consisting of all inner automorphisms $x \longrightarrow uxu^{-1}$, $x \in \Lambda$, where u ranges over all units of Λ. We set

$$(9.10) \qquad \begin{cases} \text{Autcent } \Lambda = \{f \in \text{Aut } \Lambda : f(c) = c \text{ for all } c \in C\}, \\ \text{Outcent } \Lambda = \text{Autcent } \Lambda/\text{In } \Lambda. \end{cases}$$

Now let M be an invertible Λ-ideal in A, and let $f \in$ Autcent Λ. We define a new bimodule M_f having the same elements as M, but with the action of Λ given by

$$\lambda_1 \circ m \circ \lambda_2 \ (\text{in } M_f) = \lambda_1 m f(\lambda_2) \ (\text{in } M).$$

For a pair of invertible Λ-ideals M, M', it turns out (see MO, (37.16)) that $M' \cong M$ as left Λ-modules if and only if $M' \cong M_f$ for some $f \in$ Autcent Λ. Furthermore (MO, (37.14)) there is a bimodule isomorphism $M \cong M_f$ if and only if $f \in$ In Λ. This readily implies that the map

$$(9.11) \qquad \omega : \text{Outcent } \Lambda \longrightarrow \text{Picent } \Lambda, \text{ given by } \omega(f) = (\Lambda_f),$$

is a monomorphism of groups.

In some cases, the map ω is an isomorphism. For example, suppose that $\Lambda_P = R_P G$ is an integral group ring. Now for each invertible Λ_P-ideal M we have $F_P M = A_P$, and M is projective as left Λ_P-module. Hence by a result of Swan (see CR, (77.14)) it follows that $M \cong \Lambda_P$ as left Λ_P-modules. Therefore when $\Lambda_P = R_P G$ we have Outcent $\Lambda_P \cong$ Picent Λ_P, and the calculation of Picent Λ_P reduces to that of Outcent Λ_P. This fact is used in Fröhlich's calculations [7] of Picard groups.

We are still faced with the question of determining Outcent Λ in practice. This can best be done by means of an alternate description of this group. Every $f \in$ Autcent Λ extends to an automorphism of A which fixes the center of A, and hence (Skolem-Noether Theorem, MO, §7d) is an inner automorphism. Now set

$$(9.12) \qquad \textit{normalizer} \text{ of } \Lambda = N(\Lambda) = \{a \in u(A) : a\Lambda a^{-1} = \Lambda\}.$$

The preceding remarks yield

$$(9.13) \qquad \text{Outcent } \Lambda \cong N(\Lambda)/u(\Lambda)\,u(FC),$$

with the isomorphism induced by mapping each $a \in N(\Lambda)$ onto the coset containing the automorphism $\lambda \longrightarrow a\lambda a^{-1}$, $\lambda \in \Lambda$.

The relation between global and local Picard groups is given by the following basic result of Fröhlich [7] (see MO, (37.28)):

http://dx.doi.org/10.1090/cbms/026/09

(9.14) THEOREM. *Let* Λ *be an* R*-order in a semisimple* F*-algebra* A, *and let* C *be the center of* Λ. *Then there is an exact sequence*

$$1 \longrightarrow \text{Picent } C \xrightarrow{\tau} \text{Picent } \Lambda \longrightarrow \sum{}^{\oplus} \text{Picent } \Lambda_P \longrightarrow 1,$$

and Picent $\Lambda_P = 1$ *a. e.*

REMARKS. The map τ is given by $\tau(L) = (L\Lambda)$ for each invertible C-ideal L in FC. The fact that Picent $\Lambda_P = 1$ a.e. can be deduced from (9.9), as follows: let Λ' be a maximal R-order in A containing Λ. Then $\Lambda_P = \Lambda'_P$ a.e., and Picent $\Lambda'_P = 1$ a.e. by (9.9). Hence also Picent $\Lambda_P = 1$ a.e.

An invertible Λ-ideal M is called *locally free* if M is in the same genus as Λ as left Λ-module, that is, $M_P \cong \Lambda_P$ as left Λ_P-modules for each prime ideal P of R. As a matter of fact, any two-sided Λ-ideal which is locally free is automatically invertible, and is also locally free as right Λ-module. The bimodule isomorphism classes (M) of locally free invertible Λ-ideals M form a subgroup $LFP(\Lambda)$ of Picent Λ; this subgroup is called the *locally free Picard group* of Λ. There are several important cases in which $LFP(\Lambda)$ coincides with Picent Λ; this holds when Λ is commutative, by (9.7(ii)). It also holds when Λ is a maximal order. More important, by Swan's Theorem (1.8) we have $LFP(\mathbf{Z}G) = \text{Picent } \mathbf{Z}G$ for each finite group G.

The remarks following (9.11) show that

$$LFP(\Lambda_P) \cong \text{Outcent } \Lambda_P \cong N(\Lambda_P)/u(\Lambda_P)u(F_PC).$$

Furthermore, (9.14) gives the exact sequence

$$1 \longrightarrow \text{Picent } C \longrightarrow LFP(\Lambda) \longrightarrow \sum{}^{\oplus} LFP(\Lambda_P) \longrightarrow 1.$$

These formulas permit the calculation of $LFP(\Lambda)$ in various cases. Let us mention a few of the results of Fröhlich [7]:

(i) If G is a p-group, then so is $\Sigma^{\oplus}$ Picent R_pG.

(ii) If G is a p-group, so is $D(C)$, where C is the center of $\mathbf{Z}G$.

(iii) If G is a dihedral group of order $2p$, with p an odd prime, then Picent $\mathbf{Z}G \cong D(C) \oplus \text{Cl } S$, where C is the center of $\mathbf{Z}G$, and $S = \text{alg. int.}\,\{Q(\omega + \omega^{-1})\}$, with $\omega = \sqrt[p]{1}$. Further, $D(C)$ is cyclic of order $(p-1)/2$.

(iv) For G the quaternion or dihedral group of order 8, we have Picent $\mathbf{Z}G \cong D(\mathbf{Z}G)$; hence Picent $\mathbf{Z}G$ is trivial for the dihedral case, and has order 2 for the quaternion case (see (6.4), (7.4)).

(v) Let I be an ideal of R, and let $A = M_n(F)$, $\Lambda = R \cdot 1 + I \cdot M_n(R)$, so Λ is a congruence order in A. Then

$$\text{Picent } \Lambda \cong \text{Cl } R \oplus PGL(n, R/I),$$

where

$$PGL(n, R/I) = GL(n, R/I)\big/u(R/I)$$

is the projective general linear group of $n \times n$ matrices over the ring R/I.

To conclude this section, we shall describe some results of Fröhlich, Reiner and Ullom [11] concerning the relation between the locally free Picard group $LFP(\Lambda)$ and the locally free class group $\text{Cl } \Lambda$ of an order Λ.

http://dx.doi.org/10.1090/cbms/026/09

(9.15) THEOREM. *There is a homomorphism*

$$\theta : LFP(\Lambda) \longrightarrow \mathrm{Cl}\,\Lambda,$$

given by $\theta(X) = [X]$ *for* $(X) \in LFP(\Lambda)$. *If the cancellation law holds for locally free* Λ*-ideals in* A, *then* $\ker\theta \cong$ Outcent Λ.

OUTLINE OF PROOF. Let X, Y be locally free invertible Λ-ideals in A, and take $X \subset \Lambda$ without loss of generality. Then there is an exact sequence of bimodules

$$0 \longrightarrow X \longrightarrow \Lambda \longrightarrow T \longrightarrow 0,$$

with T an R-torsion module. Since ${}_\Lambda Y$ is projective, we obtain an exact sequence

$$0 \longrightarrow X \otimes Y \longrightarrow Y \longrightarrow T \otimes Y \longrightarrow 0,$$

where $\otimes$ means $\otimes_\Lambda$. But $T \otimes Y \cong T$ since Y is locally free. Comparing these two sequences and using Schanuel's Lemma, we obtain a left Λ-isomorphism $X \dotplus Y \cong \Lambda \dotplus X \otimes Y$. Further, $X \otimes Y \cong XY$, and therefore $[X] + [Y] = [XY]$ in $\mathrm{Cl}\,\Lambda$. This proves that θ is a homomorphism. Finally, suppose that $\theta(X) = 0$, so $[X] = 0$ in $\mathrm{Cl}\,\Lambda$. Assuming cancellation, we get $X \cong \Lambda$ as left Λ-modules, whence $(X) = (\Lambda_f)$ for some $f \in$ Autcent Λ by the remarks before (9.11). Thus $\ker\theta = \operatorname{im}\omega$, as claimed.

Suppose that Λ is commutative. We have already remarked that $LFP(\Lambda) =$ Picent Λ in this case. Further, it is clear that the map θ in (9.15) is a surjection. Thus we obtain Picent $\Lambda = LFP(\Lambda) \cong \mathrm{Cl}\,\Lambda$ whenever Λ is commutative. Thus, in this case the method of "adding" locally free ideals used to define the class group $\mathrm{Cl}\,\Lambda$ in §1 turns out to be equivalent (up to isomorphism) to the method of multiplying locally free ideals when defining the Picard group Picent Λ.

Let us next obtain an explicit formula for $\operatorname{cok}\theta$ in terms of ideles. We define the *idele normalizer* of Λ by

$$\bar{N}(\Lambda) = \left\{ (x_p) \in \prod N(\Lambda_p) : x_p \in u(\Lambda_p) \ \text{a.e.} \right\}.$$

Each $(X) \in LFP(\Lambda)$ is then expressible as $X = A \cap \{\bigcap \Lambda_p x_p\}$ for some $(x_p) \in \bar{N}(\Lambda)$. This gives a surjection of $\bar{N}(\Lambda)$ onto $LFP(\Lambda)$, and yields an isomorphism

$$\phi^* : \bar{N}(\Lambda)/u(\Lambda)u(FC) \cong LFP(\Lambda).$$

On the other hand, each $x \in \bar{N}(\Lambda)$ determines an element $\theta^*(x) \in JK(A)$. The homomorphism θ^* induces a map

$$\theta^* : \bar{N}(\Lambda)/u(\Lambda)u(FC) \longrightarrow JK(A)/\operatorname{im} K_1(A) \cdot \operatorname{im} UK(\Lambda),$$

and we obtain a commutative diagram

$$\begin{array}{ccc} \bar{N}(\Lambda)/u(\Lambda)u(FC) & \xrightarrow{\ \theta^*\ } & JK(A)/\operatorname{im} K_1(A) \cdot \operatorname{im} UK(\Lambda) \\ \phi^* \big\downarrow & & \phi \big\downarrow \\ LFP(\Lambda) & \xrightarrow[\ \theta\ ]{} & \mathrm{Cl}\,\Lambda. \end{array}$$

Therefore $\operatorname{cok}\theta \cong \operatorname{cok}\theta^*$.

For convenience, we restrict our attention to the case where F is an algebraic number field. We may then use the formula for $\mathrm{Cl}\,\Lambda$ given in (2.21), and calculate $\operatorname{cok}\theta^*$ as a quotient of $I(FC)/(FC)^+ \cdot \prod nr\, u(\Lambda_p)$. We obtain

http://dx.doi.org/10.1090/cbms/026/09

(9.16) THEOREM. *Let us put*

$$nr\,\bar{N}(\Lambda) = \{(nr\,x_P) \in J(FC) : (x_P) \in \bar{N}(\Lambda)\}.$$

Then

$$\operatorname{cok}\theta \cong J(FC)/(FC)^+ \cdot nr\,\bar{N}(\Lambda) \tag{9.17}$$

if F is an algebraic number field.

Formula (9.17) may be restated in terms of ideals, rather than ideles; for the explicit statement, see [11, Theorem 5.13]. An interesting consequence of the ideal-theoretic version is as follows:

(9.18) COROLLARY. *Keeping the notation of* (2.17), *let* $(A_i : F_i) = m_i^2$, $1 \leq i \leq s$. *Then the exponent of* $\operatorname{cok}\theta$ *divides the least common multiple of* $m_1, \ldots, m_s$.

Finally, assume that stable isomorphism implies isomorphism for locally free Λ-lattices, and let $\Gamma = M_n(R) \otimes_R \Lambda \cong M_n(\Lambda)$. Then there is a commutative diagram with exact rows

$$\begin{array}{ccccccc} 1 & \longrightarrow & \text{Outcent}\,\Lambda & \longrightarrow & LFP(\Lambda) & \longrightarrow & \text{Cl}\,\Lambda \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ 1 & \longrightarrow & \text{Outcent}\,\Gamma & \longrightarrow & LFP(\Gamma) & \longrightarrow & \text{Cl}\,\Gamma, \end{array}$$

where $\alpha(f) = 1 \otimes f$, and β, γ are "change of rings" maps. As shown in [11],

$$\ker\gamma = \{x \in \text{Cl}\,\Lambda : nx = 0\}, \quad \operatorname{cok}\gamma \cong \text{Cl}\,\Lambda/n \cdot \text{Cl}\,\Lambda.$$

In particular, if Λ is a commutative R-order, it follows that

$$\text{Outcent}\,M_n(\Lambda) \cong \{x \in \text{Cl}\,\Lambda : nx = 0\}.$$

Thus

$$\text{Outcent}\,M_n(R) \cong \{x \in \text{Cl}\,R : nx = 0\}.$$

Therefore for each automorphism f of $M_n(R)$ fixing its center, f^n is an inner automorphism of $M_n(R)$; this result was first proved by Rosenberg and Zelinsky [36a].

The list of references which follows includes almost all articles on class groups and Picard groups which have appeared up to this time, or are available in preprint form. It also includes a number of basic texts for further reading about the various topics considered in these lectures. These texts, arranged by subject matter, are:

Homological algebra: [14], [37],
Algebraic K-theory: [1], [20], [23], [40a], [41],
Representation theory: [3], [5a], [14a],
Integral representations: [3], [26], [34], [35].

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Index

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