Conference Board of the Mathematical Sciences

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Regional Conference Series in Mathematics

Number 28

Lectures on Hilbert Cube Manifolds

T. A. Chapman

American Mathematical Society
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Lectures on Hilbert Cube Manifolds

The goal of these lectures is to present an introduction to the geometric topology of the Hilbert cube Q and separable metric manifolds modeled on Q, which we call *Hilbert cube manifolds* or *Q-manifolds.* In the past ten years there has been a great deal of research on Q and Q-manifolds which is scattered throughout several papers in the literature. We present here a self-contained treatment of only a few of these results in the hope that it will stimulate further interest in this area. No new material is presented here and no attempt has been made to be complete. For example we have omitted the important theorem of Schori-West stating that the hyperspace of closed subsets of $[0,1]$ is homeomorphic to Q. In an appendix (prepared independently by R. D. Anderson, D. W. Curtis, R. Schori and G. Kozlowski) there is a list of problems which are of current interest. This includes problems on Q-manifolds as well as manifolds modeled on various linear spaces. We refer the reader to this for a much broader perspective of the field.

In some vague sense Q-manifold theory seems to be a "stable" PL n-manifold theory. This becomes more precise in light of the Triangulation and Classification theorems of Chapters XI and XII. In particular, all handles can be straightened and consequently all Q-manifolds can be triangulated. Thus there are delicate finite-dimensional obstructions which do not appear in Q-manifold theory. This is perhaps why the proofs of the topological invariance of Whitehead torsion (Chapter XII) and the finiteness of homotopy types of compact ANRs (Chapter XIV) first surfaced at the Q-manifold level.

In the first four chapters we present the basic tools which are needed in all of the remaining chapters. Beyond this there seem to be at least two possible courses of action. The reader who is interested only in the triangulation and classification of Q-manifolds should read straight through (avoiding only Chapter VI). In particular the topological invariance of Whitehead torsion appears in §38. The reader who is interested in R. D. Edwards' recent proof that every ANR is a Q-manifold factor should read the first four chapters and then (with the single exception of 26.1) skip over to Chapters XIII and XIV.

These lectures were delivered in October, 1975, at Guilford College as part of the Regional Conference Program sponsored by the Conference Board of the Mathematical Sciences with the support of the National Science Foundation. I wish to express my appreciation to the Conference

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T. A. Chapman

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APPENDIX

OPEN PROBLEMS IN INFINITE-DIMENSIONAL TOPOLOGY

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I. Introduction

This problem list is a successor to earlier problem lists prepared following conferences in Ithaca (January 1969), Baton Rouge (December 1969), Oberwolfach (September 1970), and Baton Rouge (October 1973). It was prepared following a special conference in Athens, Georgia (August 1975) and the CBMS regional conference in Greensboro (October 1975). Earlier lists were published as Mathematisch Centrum Report ZW 1/71 and as part of Mathematical Centre Tract 52, 1974, pp. 141-175, both from Amsterdam.

The present list is not, of course, a complete list of all open questions known to the conference participants, but it does include representative problems from most of the principal areas of current activity in the point-set topology of infinite-dimensional spaces and manifolds known to the authors. Space considerations have forced the omissions of some topics included in earlier lists such as uniformly continuous and

Lipschitz homeomorphisms in which the recent activity has been by others (Aharoni, Enflo, Mankiewicz) than the participants. This problem list has been organized by R.D. Anderson, D.W. Curtis, G. Kozlowski and R.M. Schori with help from many others. An effort has been made to bring the past lists up-to-date and to recognize the current, as distinct from the earlier, areas of interest.

It is recognized that a few of the problems listed below may be inadequately worded, be trivial or be known. Because of many interrelationships, some aspects of various problems are listed under more than one heading.

The following mathematicians (with addresses listed in the AMS-MAA Combined Membership List) took part in the problem sessions and are sources of continuing information on many of the problems: R.D. Anderson, T.A. Chapman, D.W. Curtis, R. Geoghegan, G. Kozlowski, R.M. Schori, and James E. West; the following participants are sources on certain types of problems: D.E. Edwards, R.D. Edwards, M. Handel, H. Hastings, W.E. Haver, R. Heisey, J.E. Keesling, J. Quinn, R.B. Sher, W.E. Terry, R.Y-T Wong, and S. Ferry.

Henryk Torunczyk of the Mathematics Institute of the Polish Academy of Sciences in Warsaw and Czeslaw Bessaga of the Mathematics Institute of the University of Warsaw are also knowledgeable about many problems and results, particularly those dealing with linear space problems.

We briefly review three of the important areas of recent results in I-D topology and then mention three areas of promising research activity. Our effort is to show the evolving relationships of I-D topology with other areas of mathematics.

1. Recent results have emphasized the close relationships of I-D topology with AR and ANR theory. Specifically, using Q-manifold theory, James E. West showed that every compact metric ANR has the homotopy type of a finite polyhedron. More recently, R.D. Edwards has shown the stronger result that a locally compact separable metric space X is a Q-manifold factor ($X \times Q$ is homeomorphic to a Q-manifold) iff X is an ANR. For compacta this implies West's result and also characterizes compact AR's as Q-factors. It complements the earlier results of Torunczyk showing that a complete separable metric space X is an ℓ_2-manifold factor iff X is an ANR. (Indeed, Torunczyk's results on ANR's are applicable to manifolds modeled on various linear metric spaces.) Clearly, in light of Edward's results, locally compact or compact AR's or ANR's can be conveniently studied by crossing with Q and using Q-manifold theory. Many properties of AR's, ANR's and of deformations of such spaces now follow automatically.

2. In late 1974, J.L. Taylor showed that there exists a cell-like map (i.e. each point inverse has trivial shape) of an I-D compactum X onto Q such that X does not

have trivial shape. Taylor used inverse limit techniques and examples of Adams and/ or Kahn. It is a corollary of Taylor's example and known results (Curtis, R. Edwards, Keesling, Kozlowski) that there exists a cell-like map of Q onto a non-AR of infinite dimension. Thus, the known results that CE maps preserve shape if the image is finite-dimensional (Anderson, Kozlowski, and, in a special case, Sher) or if domain and range are ANR's (Armentrout-Price, Kozlowski, Lacher) cannot be totally extended. On the basis of results of Kozlowski on hereditary shape equivalences, the following questions (older, in some form) now seem more important (and possibly doable?) and are related.

(A) Does there exist a cell-like map of a finite dimensional compactum onto an I-D compactum?

(B) Does every I-D compactum contain an n-dimensional subset (not necessarily closed) for each n?

An affirmative solution to either gives a negative solution to the other. (Note that Henderson's and other examples give infinite-dimensional compacta containing no 1-dimensional subcompacta. The question about I-D compacta necessarily containing n-dimensional nonclosed subsets apparently is still open.)

3. The earlier results of Geoghegan that the space H(M) of homeomorphisms of a finite-dimensional compact n-manifold M admits an ℓ_2-factor $(H(M) \approx H(M) \times \ell_2)$ and of Torunczyk that for any complete metric ANR X, $X \times \ell_2$ is an ℓ_2-manifold immediately imply that H(M) is an ℓ_2-manifold iff H(M) is an ANR. Indeed, a recent observation by Haver is that H(M) is an ANR if for any n-cell, B^n, the space $H_\partial(B^n)$ of homeomorphisms of B^n fixed on the boundary is an AR. Thus, the problem is now reduced to an AR problem on a single n-cell. The results are known for $n = 1,2$.

4. There are a growing body of results relating the topology of Q-manifolds with the topology of finite-dimensional manifolds via the stabilization process, multiplying by or factoring out Q. The recent delicate Chapman-Siebenmann results on compactifications of Q-manifolds are a case in point. In his dissertation, Cerin has begun the investigation of the relationships between shape theory for locally compact spaces (ANR's) with Q-manifold theory exploiting the Chapman characterization of shape of compacta in terms of Z-set complements in Q. In general, there appears to be a growing area for study, both of related finite and I-D manifold properties and of shape and Q-manifold properties.

5. One of the areas of much activity and, so far, few definitive results, is that of group action on Q or Q-manifolds. The problems start with the question as to whether any two semi-free periodic actions on Q (with the same period and a single

fixed point) are equivalent and extend to questions of group actions on manifolds. The earlier result of Wong that two finite periodic actions on Q with a single fixed point are equivalent if their periods agree and if each has arbitrarily small invariant contractible neighborhoods of the fixed point remains the standard result (although somewhat more general conditions have been given by Edwards and Hastings). Delicate questions of proper homotopy type get involved.

6. The classical questions of topological characterizations of Q and s in terms other than as products, linear spaces or as convex subsets of linear spaces remain open. It seems likely that imaginative new, usable characterizations of s or Q should lead to new theories.

Finally we mention an area which recent results have pretty much finished. An early question in I-D topology concerned the union of two sets X_1 and X_2 such that X_1, X_2 and $X_1 \cap X_2$ were all homeomorphic to Q or were all homeomorphic to s. Is $X_1 \cup X_2$ necessarily homeomorphic to Q (or to s)? Similar questions existed where X_1, X_2 and $X_1 \cup X_2$ were all assumed homeomorphic to Q (or s) and the question was whether $X_1 \cap X_2$ was homeomorphic to Q (or s). All of these questions have negative answers (some rather easily). The final result to be obtained is due to Sher who used Eaton's generalized dog-bone construction applied to Q and showed that on slicing it in two we get two copies of Q whose intersection is a copy of Q but whose union is not. Complementing this, Handel has shown that a union of two copies of Q must be a copy of Q if the intersection is a Z-set copy of Q in either. Lastly, in contrast with this, Quinn and Wong have shown that the union of two convex Hilbert cubes in ℓ_2 is a cube, provided their intersection is a cube.

II. CE Images of ANR's and Q-manifolds

In the theory of Q-manifolds the foundational problem involving CE mappings is to give conditions under which the image Y under a CE map $f: M \to Y$ of a Q-manifold is homeomorphic to M. By Chapman's CE Mapping Theorem the question of Y being homeomorphic to M is equivalent to that of Y being a Q-manifold.

(M) Under what conditions do the CE images of Q-manifolds remain Q-manifolds?

In connection with problems of type (M) we shall deal only with situations in which each point-inverse is a Z-set, since shrinking out a wild arc (Wong) or a cut slice in Q produces a non-Q-manifold. Even this improved situation is circumscribed by counter-examples. A modification of Eaton's argument for the existence of dog-bone decompositions for higher dimensional Euclidean spaces shows that there is a dog-bone decomposition of Q, i.e., a surjection $f: Q \to Y$ such that Y is not Q, each nondegenerate point-inverse is a Z-set arc, and the nondegeneracy set of f is a Cantor set in Y. In this case by (1c) below Y is an AR, but even this information is not obtained in general, for Taylor's example gives a CE map $f: X \to Q$ onto the Hilbert cube which is not a shape equivalence; and considering X as a Z-set of Q and taking the adjunction $F: Q \to Q \cup_f Q = Y$ produces a CE map of Q onto a non AR.

Because of Taylor's example there is interest in finding conditions to insure that the CE image of a Q-manifold is an ANR. This problem is equivalent to the older problem of finding conditions under which the CE image of a (locally compact) ANR is an ANR. In fact, if $f: X \to Y$ is a CE map of such an ANR, then $f' = fp: X \times Q \to Y$ is a CE map (where $p: X \times Q \to X$ is the standard projection) of a Q-manifold by Edwards' result. Kozlowski defines an hereditary shape equivalence to be a proper map $f: X \to Y$ such that $f: f^{-1}B \to B$ is a shape equivalence for every closed subset B of Y, and has shown:

(1) A CE map $f: X \to Y$, with X an ANR, is an hereditary shape equivalence if and only if Y is an ANR. His Vietoris theorems then imply that Y is an ANR in the following cases:

(1a) Y is a countable union of closed finite-dimensional subspaces;

(1b) Y is compact and countable-dimensional;

(1c) the nondegeneracy set $\{y \in Y: f^{-1}(y)$ is nondegenerate$\}$ of f is finite dimensional (or more generally, is contained in a subset of Y having large inductive transfinite dimension).

(A) Under what conditions do CE images of ANR's remain ANR's?

This problem can be closely tied to (M). Say that a map $f: X \to Y$ *is determined on the subset* A of X if every nondegenerate point-inverse of f lies in A. The following useful result was known to West and others.

(2) If $f: M \to Y$ is a CE map of a Q-manifold M onto an ANR which is determined on a Z-set of M, then f is a near homeomorphism.

Now consider a CE map $f: M \to Y$ defined on a Q-manifold M. Identify $M \equiv M \times 0 \subset M \times Q$ and consider the adjunction $F: M \times Q \to (M \times Q) \cup_f Y = N$. It is a classical result of Borsuk-Whitehead-Hanner that N is an ANR iff Y is. Alternatively, identify $M \equiv M \times 1 \subset M \times [0,1]$ and consider the adjunction $M \times [0,1] \to (M \times [0,1]) \cup_f Y = M(f)$, where M(f) is the mapping cylinder of f; since M is a Z-set in $M \times Q$ and $M \times [0,1]$, these situations are related by the Z-set homeomorphism extension theorem as in the diagram

$$\begin{array}{ccc} M \times Q & \cong & M \times [0,1] \\ \downarrow & & \downarrow \\ N & \cong & M(f) \end{array}$$

where the homeomorphism is the identity on Y. By Theorem 2, N is a Q-manifold iff N is an ANR. Recapitulating, for a CE map $f: X \to Y$ one has Y an ANR iff f is an hereditary shape equivalence iff the induced map $M \times Q \to N$ is a near homeomorphism. Thus (A) is equivalent to

(Z) Under what conditions do CE images of Q-manifolds determined on Z-sets remain Q-manifolds.

The following problems stem from (A).

(CE 1) Let $f: Q \to Y$ be a surjection with each point-inverse a copy of Q. Is Y an AR? It is readily seen, using Edwards' Q-factor theorem and a cone construction, that this is equivalent to a question of Borsuk: If X is a compact ANR and $f: X \to Y$ has AR's for point-inverses, is Y an ANR?

(CE 2) Let $f: B^n \to Y$ be a CE map. Is Y an AR? This is equivalent to the question: Is the CE image of a finite-dimensional compactum finite dimensional? (The procedure, given a CE map $f: X \to Y$ with $\dim X < \infty$, is to imbed X in some B^n and consider the quotient map $F: B^n \to B^n \cup_f Y$. If $B^n \cup_f Y$ is an AR, then F and f are hereditary shape equivalences, and these do not raise dimension. The converse follows from (1a) above.

Even the following special case is of interest.

(CE 3) If it is further assumed in (CE 2) that the nondegenerate point-inverses of f are arcs, is Y an AR?

Some of these questions originated in part from the study of decompositions of manifolds. The situation at present is that one has complete information regarding the homotopy groups but no information about homotopy.

(CE 4) If X is B^n or R^n and $f: X \to Y$ is a cell-like map, is Y contractible?

A corollary of Theorems (1), (1c) and (2) is a theorem claimed earlier by Anderson.

(3) If M is a Q-manifold and $f: M \to Y$ is a CE map determined on a Z-set and whose nondegeneracy set is finite dimensional, then f is a near homeomorphism.

In the dog-bone decomposition of Q the union of the nondegenerate point-inverses does not lie even in a σ-Z-set.

(CE 5) Suppose $f: Q \to Y$ is a CE map onto an AR, with the union of the nondegenerate point-inverses lying in a σ-Z-set in Q. Is $Y \cong Q$? Suppose we require only that the union of the nondegenerate point-inverses lie in a pseudo-interior of Q?

Even cases in which there are only countably many nondegenerate point-inverses of f are of interest. [In this case (1c) shows that Y is an ANR.]

(CE 6) More specifically, if the set of nondegenerate point-inverses is countable, is $Y \cong Q$? (This is known if the union of the nondegenerate point-inverses is a G_δ.)

(CE 7) Is $Y \cong Q$, if (a) the collection of nondegenerate point-inverses is null (i.e., there are only finitely many such sets of diameter $> \epsilon$ for every $\epsilon > 0$), and/or if (b) the closure in Y of the nondegeneracy set is zero-dimensional?

(CE 8) Suppose $f: Q \to Y$ is a CE map of Q onto an AR. Find conditions and examples when $Y \times I^k \cong Q$ for some finite k. Recall that Bryant-Chapman have shown that $Y \times I \cong Q$ in the case that f has exactly one nontrivial point inverse and it is a finite-dimensional cell. R. Edwards claims that $Y \times I \cong Q$ in the case that f has exactly one nontrivial point inverse and it is a finite-dimensional cell-like compactum (this uses a variation of an argument of Stanko).

The following question relating to ℓ_2-manifolds is also of interest.

(CE 9) Suppose Y is a complete separable metric ANR which is a closed cell-like image of an ℓ_2-manifold M. Is Y in fact an ℓ_2-manifold? This amounts to asking whether the decomposition of M induced by the (compact) point inverses of the map $M \to Y$ is shrinkable in the sense of Bing. It is a consequence of known theorems that the stabilized decomposition obtained by crossing with ℓ_2 or Q is shrinkable.

III. SC Shape of Compacta in I-D Topology

Shape theory has become a useful tool in I-D topology and geometric topology. Since Chapman's proof that two Z-sets in Q have the same shape iff their complements are homeomorphic, deep results of Chapman, R. Edwards, Miller, and West involving CE maps have decisively demonstrated the power of shape theoretic concepts in solving classical problems. The problems appearing here are, for the most part, concerned only with aspects of shape theory which bear somewhat directly on I-D topology. For more purely shape theoretic problems, one can consult Borsuk's work, in which lists of problems frequently appear.

We restate two problems from the CE section.

(SC 1) Is it true that cell-like maps do not raise dimension? This question is equivalent to the following: is a cell-like map defined on a finite-dimensional space a shape equivalence?

(SC 2) Does every infinite-dimensional compactum contain subsets of arbitrarily high finite dimension? [A positive answer implies a positive answer to (SC 1).]

(SC 3) If $f: X \to Y$ is a CE map, whose nondegeneracy set is countable-dimensional, is f a shape equivalence?

(SC 4) If M and N are compact Q-manifolds and if there exist a compactum X and hereditary shape equivalences $f: X \to M$, $g: X \to N$, are M and N homeomorphic?

(SC 5) If M and N are compact Q-manifolds and if there exist a compactum

X and CE maps $f: M \to X$, $g: N \to X$, are M and N homeomorphic?

(SC 6) Give conditions so that for an inclusion $Y \to X$ which is also a shape equivalence the following condition holds whenever $X \subset P$ (an ANR): for every neighborhood V of Y and every neighborhood U of X, there is a homotopy $f_t: X \to U$, $0 \leq t \leq 1$ such that: (1) $f_0(x) = x$ for all $x \in X$, (2) $f_1(X) \subset V$, and (3) $f_t(y) = y$ for all $y \in Y$ and $0 \leq t \leq 1$.

(SC 7) If X, Y are compact metric with respective compact subsets A, B and if $f: X \to Y$ restricts to a shape equivalence $A \to B$ and maps $X \setminus A$ homeomorphically onto $Y \setminus B$, is f a shape equivalence? What if X and Y are compact ANR's?

(SC 8) If X and Y are shape equivalent UV^1-compacta, does there exist a finite diagram $X = X_0 \leftrightarrow X_1 \leftrightarrow \dots \leftrightarrow X_n = Y$ in which $X_i \leftrightarrow X_{i+1}$ is an hereditary shape equivalence either from X_i to X_{i+1} or from X_{i+1} to X_i?

(SC 9) If A is a Z-set in Q and Q/A is an ANR, is A shape-dominated by a complex?

(SC 10) Let $f: (X, x) \to (Y, y)$ be a morphism of pointed shape theory which induces an isomorphism on each homotopy pro-group. Under what hypotheses is f a (pointed) shape equivalence? (Remark: There is considerable recent literature on this problem. There are good positive theorems and good counterexamples. We are asking for the best possible positive theorem.)

IV. PF Products and Factors

R. D. Edwards has recently proved that every locally compact separable metric ANR is a Q-manifold factor. (Trivially, the converse is true.) This long-sought-after result is the Q-analogue of Torunczyk's characterization of ℓ_2-manifold factors: every complete separable metric ANR is an ℓ_2-manifold factor.

Edwards' result, together with Chapman's Q-manifold triangulation theorem, provides another proof of West's theorem that every compact metric ANR has finite homotopy type.

(PF 1) Give an internal characterization of spaces X for which $X \times I \cong Q$. What is the relationship between the conditions $X \times I \cong Q$ and $X \times X \cong Q$? Cerin has shown, extending unpublished results of Bryant and Chapman, that $(Q/A) \times I \cong Q \cong Q/A \times Q/A$ for every closed n-cell A in Q.

Another consequence of Edwards' result with a theorem of West is that every countable infinite product of nondegenerate compact metric AR's is homeomorphic to Q. The corresponding question for products of ℓ_2-factors remains open.

(PF 2) Is every countable infinite product of noncompact complete separable metric AR's homeomorphic to ℓ_2? We may suppose that each factor contains a closed copy of the line, since it is not hard to show that each product of two such factors has this property. What about the special case where each factor is a contractible Q-manifold?

As another special case we consider the following product:

(PF 3) Let P be the set of points in the closed unit ball of ℓ_2 which have at most one nonzero coordinate. (P is sometimes called the "porcupine".) Is $\Pi_{i=1}^{\infty} P \cong \ell_2$?

(PF 4) (i) Does $X \times X \cong s \Rightarrow X \cong s$?

(ii) Does $X \times I \cong s \Rightarrow X \cong s$?

(iii) Does $X \times Q \cong s \Rightarrow X \cong s$?

Note that if $X \times Y \cong s$ for some locally compact Y, then $X \times Y \times Q \cong s$, and since $Y \times Q$ is a Q-manifold, $X \times Q$ is a contractible s-manifold, hence $X \times Q \cong s$.

V. Qs Cone Characterizations of Q and s

A major continuing problem is to get useful and simple topological characterizations of Q or s that do not depend explicitly on linear space properties or on the product structure of the space. For linear space and product structures we note that Keller proved in 1931 that any compact convex infinite-dimensional subset of ℓ_2 is homeomorphic to Q, and recent work of West and Edwards combine to show that any countable infinite product of metric compacta is homeomorphic to Q iff each factor is an AR and infinitely many are nondegenerate. Results for s due to Anderson and Kadec characterize s as homeomorphic to any separable infinite-dimensional Fréchet space, and results due to Torunczyk state that $X \times s \cong s$ iff X is a topologically complete separable metric AR. The final possible characterization of s as a product of AR's is still not settled and is stated in section PF.

It would be very nice to have topological characterizations of s and Q in more general or at least different terms. Specific problems which appear interesting are the following:

(Qs 1) Are the one-point set and Q the only homogeneous contractible metrizable compacta?

(Qs 2) Is s the only homogeneous separable contractible nonlocally compact completely metrizable space? Special cases of (Qs 1) and (Qs 2) are (Qs 3) and (Qs 4) below in which contractibility is strengthened to "being a cone". Of course, homogeneity and cone structure give conditions quite similar to those of product structure.

(Qs 3) Let X be compact metric, homogeneous and homeomorphic to its own cone. Is X homeomorphic to Q?

Note. By a theorem of Schori, $\text{cone}(Y) \times I \cong \text{cone}(\text{cone}(Y))$ for any compact

Hausdorff space Y. Therefore $X \cong X \times I$. If we can prove that the projection map $p: X \times I \to X$ is either tiltable (in the sense of West) or a near-homeomorphism, then by an inverse-limit argument it follows that $X \times Q \cong X$. Furthermore, de Groot observed that X is locally homogeneous, i.e., every point $x \in X$ has arbitrarily small neighborhoods O_x such that for any two points $y, z \in O_x$, y can be mapped onto z by an autohomeomorphism of X that is the identity outside O_x; and Kroonenberg observed that X is n-point order-preserving homogeneous for any n. A possible counterexample might be obtained in the following way: Schori showed that $\text{cone}(Y) \times Q$ is homeomorphic to its own cone for every compact metric space Y. However, homogeneity and local contractibility at the cone point rule out spaces $\text{cone}(Y) \times Q$ for Y a space like the Cantor set or the universal curve.

(Qs 4) We can pose a problem similar to (Qs 3) about s. If X is homogeneous, separable, complete metric and not locally compact and $X \cong \text{cone}(X)$ (where an appropriate metric definition of cone is used), then is $X \cong s$?

VI. H Hyperspaces

The original hyperspace problems in infinite-dimensional topology have, in general, been solved. Some problems of current interest concern pseudo-interiors for hyperspaces, convex hyperspaces, and hyperspaces of noncompact spaces.

For X a metric space, 2^X denotes the hyperspace of nonempty compact subsets of X, and C(X) the hyperspace of nonempty subcontinua of X, topologized with the Hausdorff metric. Schori and West showed that $2^I \cong Q$, and more generally, that $2^\Gamma \cong Q$ for every nondegenerate finite connected graph Γ. West also showed that the hyperspace of subcontinua C(D) of a dendron D is homeomorphic to Q if and only if the branch points of D are dense. Using these results, Curtis and Schori subsequently showed that $2^X \cong Q$ if and only if X is a nondegenerate Peano continuum, and $C(X) \cong Q$ if and only if the Peano continuum X contains no free arcs.

Further results on various subspaces of 2^X, where X is a nondegenerate Peano continuum have been obtained. In particular, for $A, A_1, \dots, A_n \in 2^X$, the containment hyperspace $2^X_A = \{F \in 2^X : F \supset A\}$ is homeomorphic to Q if and only if $A \neq X$, while the intersection hyperspace $2^X(A_1, \dots, A_n) = \{F \in 2^X : F \cap A_i \neq \emptyset \text{ for each } i\}$ is always homeomorphic to Q. Also, for every compact connected polyhedron K, there exists a hyperspace $2^K_{sst} \subset 2^K$ of "small" subsets of K such that $2^K_{sst} \cong K \times Q$.

If X has an affine structure, we may consider the convex hyperspace $cc(X) \subset 2^X$ of compact convex subsets of X. Nadler, Quinn, and Stavrakas have shown that $cc(X) \cong Q$ for every compact convex subset X of ℓ^2 with $\dim X > 1$. They also show that for $X \subset R^2$ with $cc(X) \cong Q$ (X not necessarily convex), X must be a 2-cell.

Curtis, Quinn, and Schori have shown that if $X \subset R^2$ is a *polyhedral* 2-cell then $cc(X) \cong Q$ if and only if X contains no singular segments (a segment $J \subset X$ is *singular* if it contains in its interior three vertices v_1, v_2, v_3 at which X is locally nonconvex, such that the side of J determined by the middle vertex v_2 is opposite that determined by v_1 and by v_3).

Curtis has recently shown that 2^X is an AR (metric) if and only if the metric space X is connected and locally continuum-connected. Two more specific hyperspace characterizations are also obtained: (1) $2^X \cong Q \setminus \{pt\}$ if and only if X is a locally compact connected locally connected noncompact metric space; (2) a topologically complete separable connected locally connected and nowhere-locally compact metric space X is imbeddable in a Peano continuum P such that 2^X is a pseudo-interior for 2^P if and only if X admits a metric with Property S.

(H 1) Is the collection of finite subsets of I an fd-cap set for 2^I?

(H 2) Let Γ be a finite connected graph. Are the collections $\{A \in 2^\Gamma : A$ is 0-dimensional$\}$ and $\{A \in 2^\Gamma : A$ is a topological Cantor set$\}$ pseudo-interiors for 2^Γ?

Note. The above problems are due to Kroonenberg, who has answered (H 2) for $\Gamma = I$.

(H 3) If $X \subset R^2$ is a 2-cell containing no singular segments, is $cc(X) \cong Q$?

(H 4) Let ℓ^∞ be the nonseparable Banach space of bounded real sequences, and let $\sim$ be the equivalence relation in 2^{ℓ^∞} of isometry between compact metric spaces. Is the quotient space $2^{\ell^\infty}/\sim \; \cong \ell^2$? (D. Edwards has obtained some basic properties of $2^{\ell^\infty}/\sim$.)

(H 5) Is $2^X \cong \ell^2$ for every topologically complete separable connected locally connected and nowhere-locally compact metric space X?

VII. QM Hilbert Cube Manifolds

The two major problems on Q-manifolds, triangulability and classification (by infinite simple homotopy type) have been solved by Chapman. Many techniques for PL manifolds can be adapted for Q-manifolds and are usually simpler in the I-D case.

(QM 1) Give a locally flat embedding of codimension 3 of one Q-manifold into another which does not have a normal bundle. Finite-dimensional examples exist. Chapman showed that an arbitrary-codimensional embedding of Q itself in a Q-manifold is flat, which result is false of course, even in codimension 1, when we replace Q by an arbitrary Q-manifold.

(QM 2) Let X be a compact Q-manifold, and $\mathcal{U}$ a finite open cover of X by contractible open subsets such that intersections of subcollections of $\mathcal{U}$ are either empty or contractible. Is X homeomorphic to $N(\mathcal{U}) \times Q$? Here $N(\mathcal{U})$ denotes the nerve of $\mathcal{U}$.

(QM 3) Given a compact Q-manifold M, does there exist $\epsilon > 0$ such that if $r: M \to N$ is a retraction onto a Q-manifold N, with $\operatorname{diam} r^{-1}(n) < \epsilon$ for each n in N, then r is close to a homeomorphism?

A proper surjection $f: M \to N$ is an *approximate fibration* (Coram-Duvall) provided that given a space X, mappings $g: X \times \{0\} \to M$ and $H: X \times I \to N$ such that $fg = H|X \times \{0\}$, and an open cover $\mathcal{U}$ of N, there exists a mapping $G: X \times I \to M$ such that G extends g and fG and H are $\mathcal{U}$-close.

(QM 4) If $f: M \to N$ is an approximate fibration of Q-manifolds, when is f close to locally trivial maps?

(QM 5) Let M, N be compact Q-manifolds and let $f, g: M \to N$ be locally trivial maps such that f is close to g. Is f homotopic to g through such maps?

(QM 6) (Fibered Stability) Let $E \to Q$ be a compact ANR fibration over the Hilbert cube such that each fiber is the Hilbert cube. Is $E \cong E \times Q$ by fiber-preserving homeomorphism? A positive answer would imply that $E \to Q$ is a trivial bundle. The answer is known to be affirmative if the base Q is replaced by a polyhedron.

VIII. CMP Compactifications

A noncompact Q-manifold M admits a *compactification* is there exists compact Q-manifold $N \supset M$ such that $N - M$ is a Z-set in N. Chapman-Siebenmann have treated this problem and have succeeded in finding general algebraic conditions for which this is true. Chapman-Ferry have gone one step further and found algebraic conditions which quarantee that the Z-set $N \setminus M$ is a Q-manifold. Also, a theorem of West has been strengthened by Torunczyk (and independently by Ferry) to read: if X is a compact metric ANR and $A \subset X$ is a Z-set such that $X \setminus A$ is a Q-manifold, then X is also a Q-manifold. In fact, if A is closed and hazy in X, then the assumption that X is an ANR is superfluous [see ANR].

Here is a question concerning a finite-dimensional version of the Chapman-Siebenmann result.

(CMP 1) If K is a noncompact polyhedron, when can we add a compactum to K to obtain an ANR?

(CMP 2) If K is a noncompact polyhedron, when is the one-point compactification of K an ANR?

(CMP 3) If $E \to S^1$ is a locally trivial bundle with fiber F, a noncompact Q-manifold, such that F admits a compactification, when does there exist a locally trivial bundle $\widetilde{E} \to S^1$ which contains E as a subbundle and such that each fiber $\widetilde{E}_x$ is a compact Q-manifold compactifying E_x.

(CMP 4) The above definitions may be interpreted in terms of ANR's. Which ANR's A admit Z-set compactifications? A necessary condition is that $Q \times A$ admits a Z-set compactification. Therefore, A is tame at ∞ in the sense of Chapman-Siebenmann and the invariants σ_∞ and τ_∞ must vanish. Does this suffice?

Equivalent question: If the Q-manifold $Q \times A$ admits a Z-set compactification, does A admit one?

IX. GA Compact Group Actions

One of the areas of greatest current interest (and frustration) in infinite-dimensional topology concerns questions of compact metric group actions on Q or on Q-manifolds. It is known by West's work that all compact groups can operate on ℓ_2 with an arbitrary closed set as the set of fixed points. As noted below, it is a routine application of covering space theory to show that every two fixed point free periodic homeomorphisms of prime period p on ℓ_2 are equivalent (i.e., conjugates of each other). Many interesting examples are known concerning actions on Q or on Q-manifolds, but basic questions are still open in this latter category.

Questions of group actions on Q or s or on manifolds modeled on them are quite different from those of finite-dimensional topology but examples from finite-dimensional manifolds and polyhedra provide a rich core of building blocks. Since each of Q and s can be represented as infinite products of copies of itself or of other factors, group actions can be induced on the product by using actions on the factors as explained in Examples 1, 2, and 3 below. We use the result due to West that any countable infinite product of nondegenerate Q-factors is homeomorphic to Q together with the result of Edwards that each compact AR is a Q-factor. Thus, any countable infinite product of compact nondegenerate AR's is homeomorphic to Q. We also use the fact that the cone of any ANR is an AR.

An action of G on X is *effective* or *free* if for each $g_1, g_2 \in G$ and $x \in X$, $g_1 x = g_2 x$ iff $g_1 = g_2$, i.e., each orbit is full. An action is *strongly semi-free* if there exists one point p fixed under all of G and G acting on $X \setminus \{p\}$ is free. Let $Q_0 = Q \setminus \{pt\}$. Since $s \cong Q \times s \cong Q_0 \times s$ then actions on Q or Q_0 can be used to induce actions on s.

Example 1. (West) Since any compact Lie group G acts strongly semi-freely on cone (G) by left translation on levels, we know that G acts strongly semi-freely on the countable infinite product Π_i (cone G) $\cong Q$ and thus freely on Q_0 and also on s. We call such action the *standard* free action on Q_0 or s.

Example 2. Let G be any strongly semi-free action on Q. Then G can be considered as acting on (cone Q) $\cong$ Q with an arc T of fixed points and with the original action on each level of the cone above the vertex. Now collapse the arc T of fixed points to a point. Since T is a Z-set, the resultant space Q' is still homeomorphic to Q but the induced action on Q' has interesting invariant sets.

Example 3. For each $i > 0$, let X_i be a nondegenerate AR and let G_i be a strongly semi-free group action on X_i. Let π_{i+1} be a homeomorphism of G_{i+1} onto G_i. Then $\varprojlim G_i$ acts strongly semi-freely on $\Pi X_i \cong Q$ and freely on Q_0 under the coordinate defined action. Thus, for example, any solenoidal group or Cantor group can act strongly semi-freely on Q. Also, regarding the bonding maps as isomorphisms we can induce many different looking actions of G_1 on Q.

Wong has shown that any two strongly semi-free period p homeomorphisms of Q which have arbitrarily small invariant contractible neighborhoods of the fixed point (are *trivial* at the fixed point) are equivalent. This is still the fundamental known result. It shows that the various semi-free period p examples stemming from Examples 2 and 3 above are equivalent.

The basic apparatus for studying free actions of finite groups on Q_0 or s is elementary covering space theory. Thus, for example, two Z_p free actions are equivalent if their orbit spaces are homeomorphic. For such actions on s or Q_0, the orbit spaces are Eilenberg-Mac Lane spaces whose homotopy types are characterized merely by their homotopy groups. Since all ℓ_2-manifolds (or Q-manifolds admitting a half-open interval factor) are characterized by homotopy type, we know that any two orbit spaces of the appropriate type are homeomorphic and hence the actions inducing the orbit spaces are equivalent. For Q_0, the argument breaks down since Q-manifolds are not, in general, characterized by homotopy types but by infinite simple homotopy type.

West used Siebenmann's work on infinite-simple-homotopy equivalences and Chapman's triangulations of Q-manifolds to show that for finite (or discrete countable) G, free G-actions on Q_0 are classified by the *proper* homotopy type of the orbit spaces. D.A. Edwards and Hastings showed that (1) a Siebenmann-type Whitehead theorem fails in the infinite-dimension situation, and (2) uniqueness of group actions fails in pro-homotopy theory (pro-H). It thus appears that some combination of homotopy theory and geometry is needed to settle the uniqueness of free G-actions on Q_0.

Some typical specific questions related to semi-free actions on Q are the following. If they have negative answers, then classification questions naturally arise.

(GA 1) For what prime $p > 2$ are every two period p homeomorphisms of Q with exactly one fixed point equivalent? A similar question about actions of nonprime period can be posed. No counterexamples are yet known.

(GA 2) Let $f: Q \to Q$ be a homeomorphism of Q onto itself with exactly one fixed point and with f of prime period p. Must f be trivial at x?

The concept of triviality can be extended to a periodic homeomorphism fixed on an arbitrary contractible closed set.

If $m: Y \to Y$ is a map, let $\varphi(m)$ denote the set of fixed points of m.

(GA 3) Suppose $f, g: Q \times [0,1] \to Q \times [0,1]$ are periodic level-preserving homeomorphisms of period p having fixed point sets $\varphi(f) = x \times [0,1] = \varphi(g)$ for some point $x \in X$. Is f equivalent to g by means of a level-preserving homeomorphism $h: Q \times [0,1] \to Q \times [0,1]$?

(GA 4) What if we assume, in addition, that both f and g are trivial at $x \times [0,1]$?

If the above questions have affirmative answers, we may consider replacing $[0,1]$ by $[0,1]^n$ or a polyhedron.

(GA 5) Let K be a Z-set in Q which is homeomorphic to $[0,1]^n$. Suppose $f, g: Q \to Q$ are period p homeomorphisms such that $\varphi(f) = K = \varphi(g)$ and both f and g are trivial at K. Is f equivalent to g?

When considering group actions on Q-manifolds as distinct from on Q itself the basic questions concern conditions under which actions are factorable into an action on a finite-dimensional manifold (or polyhedron) by an action on Q? And, if two actions are free and factorable then we would like to know conditions under which they are equivalent [see West (preprint) for examples of nonequivalence]. If there is an action which is not factorable, then it involves an essential mixing of finite and infinite-dimensional phenomena which would be interesting to identify. Note that solenodial or Cantor group actions on Q itself are essentially I-D type actions.

Specifically we ask:

(GA 6) Under what conditions can a compact group action G on a Q-manifold M regarded as $K \times Q$ for some polyhedron K be factored into an action on K by an action on Q? (There are actions on $S^1 \times Q$ which arise from maps rather than group actions.)

(GA 7) Under what conditions (on M?) are two free Z_p-actions on M necessarily equivalent?

X. TD Topological Dynamics

There has so far been practically no study of flows on Hilbert cube manifolds but many natural questions arise. Since $S^1 \times Q$ is homeomorphic to $([0,1] \times Q)/R$ for any homeomorphic identification R of $\{0\} \times Q$ with $\{1\} \times Q$, any discrete flow on Q can be canonically imbedded in a continuous flow on $S^1 \times Q$. Questions of the existence of minimal sets and of various types of flows such as expansive flows have not yet been studied beyond fairly obvious examples. It is not hard to show that Q itself admits a regularly almost periodic homeomorphism which is not periodic. Also, as a countable infinite product of itself, Q admits a shift homeomorphism.

We list two special problems as representative of the much wider class of open problems inherent in the types of flows studied in topological dynamics.

(TD 1) (a) Is $S^1 \times Q$ a minimal set, i.e., does $S^1 \times Q$ admit a discrete flow with all orbits being dense?

Clearly, such a flow cannot be described coordinatewise as a flow on S^1 cross a flow on Q since any discrete flow on Q has a fixed point and thus any composite flow would have an invariant circle. An affirmative answer to (TD 1)(a) would thus require a flow that does more than mix a flow on a finite-dimensional manifold or complex with one on Q itself.

(b) A more general question can be asked: Is any compact Q-manifold a minimal set?

(TD 2)(a) Does Q admit an expansive flow, i.e., is there a homeomorphism $h: Q \to Q$ and a number $\epsilon > 0$ such that for each $x, y \in Q$, $x \neq y$, there is an $n, -\infty < n < \infty$, for which $d(h^n(x), h^n(y)) > \epsilon$?

It seems likely that the answer is negative since interesting flows on Q involve some switching of coordinates and geometrically all high indexed coordinate spaces are of small diameter. Thus, any two points must be "spread apart" during the short time they are distinguished in only a few coordinates.

(b) A more general question can be asked: Does any compact Q-manifold admit an expansive flow?

XI. M Manifolds Modeled on Infinite-Dimensional Linear Spaces

The basic classification and representation theorems for manifolds modeled on many of the infinite-dimensional linear spaces were largely done in the late 1960's by Anderson and Schori, Henderson, West, and Chapman and supplemented by results from infinite-dimensional differential topology.

We quote these theorems only in the ℓ_2-manifold case. They are as follows:

(1) Two ℓ_2-manifolds are homeomorphic iff they are of the same homotopy type.

(2) M is an ℓ_2-manifold iff $M \cong K \times \ell_2$ where K is a countable, locally-finite simplicial complex.

(3) If M is an ℓ_2-manifold, then M can be embedded as an open subset of ℓ_2.

In the early 1970's, Torunczyk gave a characterization of ℓ_2-manifold factors.

(4) If X is a separable complete metric space, then $X \times \ell_2$ is an ℓ_2-manifold iff X is an ANR.

These theorems motivated much of the later work on Q-manifolds. The following problems are open. (See also (CE 9).)

(M 1) Let X be a topologically complete separable metric space.

(i) If X is an ANR, $Y \subset X$ is dense in X, and Y is an ℓ_2-manifold, under what conditions can we conclude that X is an ℓ_2-manifold? Torunczyk has proved this result in the case that $X \setminus Y$ is a Z-set in X (recall that Z-sets are closed and thus Y is open in X).

(ii) Let M be an ℓ_2-manifold, and suppose that $X \subset M$ is the closure of an open set Y. Under what conditions can we conclude that X is an ℓ_2-manifold?

Henderson has observed relative to (i) that if Z-sets are strongly negligible in X and if $X \setminus Y$ is a countable union of Z-sets, then $X \cong Y$. However, it seems difficult to verify these conditions in many naturally arising cases.

(M 2) For M a separable ℓ_2-manifold, can every homeomorphism of M onto itself be approximated by diffeomorphisms? Burghelea and Henderson have proved that such homeomorphisms are isotopic to diffeomorphisms.

In the following three problems we assume K and M to be ℓ_2-manifolds and K to be a closed subset of M. Then K is said to have *local deficiency* n *at a point* p if there exist an open set U with $p \in U$ and a homeomorphism h of $(-1,1)^n \times \ell_2$ onto U such that $h(\{0\} \times \ell_2) = K \cap U$. If K has local deficiency n at every point of K, then we say that K has local deficiency n. Let $R \subset K$ be such that (a) R consists of a single point, (b) R is compact, or (c) R is a Z-set in M and a Z-set in K.

(M 3) If K has local deficiency 1 at every point of $K \setminus R$, does K have local deficiency 1 for cases (a), (b) and (c) above?

(M 4) For $n > 1$, under what conditions does local deficiency n at every point of $K \setminus R$ imply that K has local deficiency n for cases (a), (b) and (c) above? Kuiper has given examples for $n = 2$ where R is a single point, an arbitrary n-cell, or a copy of ℓ_2, such that K does not have local deficiency 2. The examples involve knots. For $n > 2$ no examples are known.

(M 5) For $n > 1$, does local deficiency n imply the existence of a neighborhood U of K such that U is the total space of a fibre bundle over K with fibre $(-1,1)^n$?

(M 6) Let M and K be ℓ_2-manifolds with $K \subset M$ and K a Z-set in M. Then K may be considered as a "boundary" of M, i.e., for any $p \in K$ there exists an open set U in M with $p \in U$ and a homeomorphism h of U onto $\ell_2 \times (0,1]$ such that $h(K \cap U) = \ell_2 \times \{1\}$. Under what conditions on the pair (M,K) does there exist a homeomorphism h of M onto ℓ_2 such that the topological boundary of h(M) in ℓ_2 is h(K)? It is known that if the identity map of K into M induces a homotopy equivalence of K and M, then the embedding is possible.

(M 7) Let $\xi : E \to B$ be a fibre bundle over a paracompact space B with fibre F an ℓ_2-manifold. Suppose K is a closed subset of E such that $K \cap \xi^{-1}(b)$ is a Z-set in each $\xi^{-1}(b)$. Is there a fibre-preserving homeomorphism of $E \setminus K$ onto E?

(M 8) Is a locally contractible complete separable metric topological group which is not locally compact an ℓ_2-manifold?

(M 9) If G is a locally contractible separable metric topological group which is the countable union of compact finite-dimensional subsets and not locally compact, then is G an ℓ_2^f-manifold?

Note. No Q-manifold supports a topological group structure.

XII. RQ R^∞ and Q^∞-manifolds

The problems here deal with adapting the basic framework of ℓ_2-manifolds to R^∞ and Q^∞-manifolds where $R^\infty = \operatorname{dir\,lim} R^n$ and $Q^\infty = \operatorname{dir\,lim} Q^n$. The results are due to Heisey.

(RQ 1) If M is an R^∞-manifold, is $M \times R^\infty \cong M$? (This is true if M is an open subset of R^∞.)

(RQ 2) For M a Q^∞-manifold, it is known that $M \times Q^\infty \cong M$. Are there homeomorphisms $h : M \times Q^\infty \to M$ arbitrarily close to the projection $M \times Q^\infty \to M$?

(RQ 3) For every Q^∞-manifold M is there a countable locally-finite simplicial complex X such that $M \cong X \times Q^\infty$? What about the case for R^∞-manifolds?

XIII. ANR Characterizations of ANR's

Incentives for finding characterizations of ANR's are provided by Edwards' and Torunczyk's results that products of ANR's with appropriate standard I-D spaces are infinite-dimensional manifolds, and the fact that certain characterizations of spaces as I-D manifolds now hinge upon whether the spaces involved are ANR's.

The following results, which give sufficient conditions for a space to be an ANR, have recently been useful.

(i) (Haver) If X is a locally contractible metric space that can be written as a countable union of finite-dimensional compacta then X is an ANR.

(ii) (Torunczyk) X is an ANR iff there is a space E such that $X \times E$ has a basis $\mathcal{B}$ of open sets such that for any finite subcollection $\mathcal{C}$ of $\mathcal{B}$, the intersection $\bigcap \mathcal{C}$ is path-connected and all its homotopy groups are trivial.

(iii) (Kozlowski) Y is an ANR if there is an ANR X and a map $f : X \to Y$ onto a dense subset of Y with the property that for every open cover $\mathcal{V}$ of Y there exist a homotopy $h_t : X \to X$ $(0 \leq t \leq 1)$ and a map $g : Y \to X$ such that $h_0 = id_X$ $h_1 = gf$, and the homotopy is limited by $f^{-1}\mathcal{V}$.

The following questions are inspired by the homeomorphism group problem.

(ANR 1) If a space has a basis of contractible open neighborhoods, is it an ANR?

(ANR 1a) If a topological group has a basis of contractible open neighborhoods, is it an ANR?

More classical are the relations of linear spaces to AR's.

(ANR 2) Is every metrizable linear space an AR?

(ANR 2a) Is every separable metrizable linear space an AR?

Problem 2a and the homeomorphism group problems have been linked to negligibility properties.

According to Kozlowski, a subset A of X is *hazy*, provided the inclusion $U \backslash A \to U$ is a homotopy equivalence for every open subset U of X. He has shown that a map $f : X \to Y$ is a homotopy equivalence over every open subset of Y if and only if f is a fine homotopy equivalence. As a corollary one has that if $X \backslash X_0$ is hazy in X and X_0 is an ANR, then X is an ANR.

At present, there seems to be substantially more difficulty in verifying that a subset is hazy rather than just l.h.n. In particular, the following questions are open.

(ANR 3) Is $X \backslash X_0$ hazy in X when

(a) X is a separable linear space and X_0 is the linear hull of a countable dense subset,

(b) X is the component of the identity in the homeomorphism group H(M) of a closed PL manifold M of dimension ≥ 5 and X_0 consists of all PL-homeomorphisms of M which are in X.

In (ANR 3a) and (3b), it is known that $X \backslash X_0$ satisfies a weaker negligibility property.

A subset A of X is said to be *locally homotopy negligible* (abbrev. l.h.n), provided that the inclusion $U \backslash A \to U$ is a weak homotopy equivalence for every open set U in X. Torunczyk has shown this to be equivalent to his original definition of l.h.n and has also shown that if X is an ANR and $X \backslash X_0$ is l.h.n., then X_0 is an ANR. Unfortunately, the converse of this last result is false: Taylor's example gives a CE map $f : Q \to Y$ such that Y is not an ANR, although Y is an l.h.n. subset of the mapping cylinder M(f) of f (Lacher, Torunczyk) and M(f) - Y is an ANR.

XIV. HS Spaces of Homeomorphisms and Mappings

Let M be a compact n-manifold; then H(M) denotes the space of homeomorphisms on M and $H_\partial(M)$ denotes the subspace of H(M) consisting of those h which are the identity on the boundary ∂M (in case $\partial M = \emptyset$, $H_\partial(M) = H(M)$). It is known (Anderson) that the space $H_\partial(I)$ is homeomorphic to s (or ℓ_2).

The following is the problem of greatest current interest involving ℓ_2-manifolds and is often referred to as the "Homeomorphism Group Problem".

(HS 1) For M a compact n-manifold, is $H_\partial(M)$ an ℓ_2-manifold?

Much work has been done on this problem so far with the major results being:

(1) (Geoghegan) For every manifold M of positive finite dimension, $H_\partial(M) \times \ell_2 \cong H_\partial(M)$.

(2) (Torunczyk) If $H_\partial(M)$ is an ANR, then $H_\partial(M) \times \ell_2$ is an ℓ_2-manifold.

As a corollary of these results we know that $H_\partial(M)$ is an ℓ_2-manifold iff $H_\partial(M)$ is an ANR. Haver has given the following reduction of $H_\partial(M)$ being an ANR to the problem of showing $H_\partial(B^n)$ is an AR: For a given compact n-manifold M obtain a cover of M by n-cells B_i^n $(1 \leq i \leq p)$; by Edwards and Kirby there is an open neighborhood N of the identity such that any $h \in N$ can be written as the composition $h = h_p \ldots h_1$, where $h_i \in H_\partial(B_i^n)$, and the assignment $h \to (h_p, \ldots, h_1)$ from N into $P = \Pi_{i=1}^p H_\partial(B_i^n)$ defines a map $\varphi : N \to P$; clearly composition defines a map of an open neighborhood G of φN into N, which establishes $\varphi N \cong N$ as a retract of G; thus, by Hanner's theorem, if $H_\partial(B^n)$ is an AR, $H_\partial(M^n)$ is an ANR.

Consequently, (HS 1) has been reduced to the following.

(HS 2) Is $H_\partial(B^n)$ $(n > 2)$ an AR?

Mason and Luke have shown that $H_\partial(M)$ is an ANR for any compact 2-manifold and hence $H_\partial(M)$ and H(M) are ℓ_2-manifolds.

Work has also been done on PLH(M), the space of piecewise-linear homeomorphisms of a compact PL manifold M. Combined work of several authors, finally explicitly stated by Keesling and Wilson, shows that PLH(M) is an ℓ_2^f-manifold. Combining this result with a theorem of Whitehead and the discussion on hazy subsets in [ANR], the Homeomorphism Group Problem for closed PL manifolds M has been reduced to the following.

(HS 3) Let M be a closed PL manifold of dimension at least 5. Is every open subset of H(M) homotopically dominated by a CW complex?

Haver has studied $\overline{H}(M)$, the closure of H(M) in the space of mappings of a compact manifold M. He has shown that $\overline{H}(M) \setminus H(M)$ is a countable union of Z-sets in $\overline{H}(M)$ and, hence, it follows that if $\overline{H}(M)$ is an ℓ_2-manifold, so is H(M).

(HS 4) Is $\overline{H}(M)$ an ANR and, hence, an ℓ_2-manifold, since Geoghegan and Henderson have shown that $\overline{H}(M) \times \ell_2 \cong \overline{H}(M)$? This would imply that $H(M)$ is an ℓ_2-manifold.

(HS 5) Can the elements of $\overline{H}(M)$ be continuously approximated by homeomorphisms, i.e., does there exist for each $\epsilon > 0$ a map $h: \overline{H}(M) \to H(M)$ such that $d(h, id) < \epsilon$?

XV. LS Linear Spaces

In a sense, infinite-dimensional topology originated with problems posed by Fréchet and by Banach concerning the topological as distinct from the joint linear and topological structure of linear spaces. While almost all of the originally posed problems have been solved, several intriguing open questions exist. Bessaga, Pełczyński and Torunczyk are probably the best sources concerning such problems. We first list problems concerning separable spaces.

(LS 1) Is every I-D separable normed space homeomorphic to some pre-Hilbert space, i.e., to a linear subspace (not necessarily closed) of a Hilbert space?

(LS 2) Let X be an I-D separable pre-Hilbert space. Is $X \times R \cong X$? $X \times X \cong X$? $X_f^\omega \cong X$ or $X^\omega \cong X$? The answers are probably negative for the added condition of uniform homeomorphisms.

(LS 3) If a σ-compact separable normed space E contains a topological copy Q' of Q, is E homeomorphic to $\{x \in \ell_2 : \sum i^2 \cdot x_i^2 < \infty\}$? Note that the closed convex hull of Q' need not be compact.

(LS 4) Identify classes of subsets of ℓ_2 which are all homeomorphic to Q. The result should be more general or in a different context than the Keller characterization of all I-D compact convex subsets of ℓ_2 as homeomorphic to Q.

(LS 5) Let E be locally convex linear metric space and let X be a noncomplete retract of E. Is $X \times E^\omega \cong E^\omega$? It is known by Torunczyk that $X \times E^\omega \times \ell_2^f \cong E^\omega \times \ell_2^f$ and that if X is complete, then $X \times E^\omega \cong E^\omega$.

Some problems on nonseparable spaces are the following.

(LS 6) Is every I-D Banach space homeomorphic to some Hilbert space?

(LS 7) For every I-D Banach space E is $E \cong E^\omega$? (The result is known for Hilbert spaces.) A positive answer to this question would extend the domain of many theorems on nonseparable spaces and manifolds which suppose $E \cong E^\omega$.

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