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PREFACE

Our knowledge about fractional parts of linear polynomials is fairly satisfactory. Dirichlet's Theorem tells us that for every α and for $N > 1$, there is a natural $n \leq N$ with $\|\alpha n\| < N^{-1}$, where $\| \cdots \|$ denotes the distance to the nearest integer; and this bound is best possible. Our knowledge about fractional parts of nonlinear polynomials is not so satisfactory. In these Notes we start out with Heilbronn's Theorem on quadratic polynomials $f(n) = \alpha n^2$, according to which there is a natural $n \leq N$ with $\|f(n)\| < N^{-1/2+\epsilon}$. From this we branch out in three directions. In §§7–12 we deal with arbitrary polynomials with constant term zero. In §§13–19 we take up simultaneous approximation of quadratic polynomials, and in §§20, 21 we discuss special quadratic polynomials in several variables. There are many open questions; in fact, most of the results obtained in these Notes are almost certainly not best possible. Since the theory is not in its final form, I have refrained from including the most general situation, i.e. simultaneous fractional parts of polynomials in several variables of arbitrary degree.[1] On the other hand, I have given all the proofs in full detail, at a leisurely pace.

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Wolfgang M. Schmidt
December 1976

[1]For further references, covering a rather wider area, see Malyshev and Podsypanin [1974].

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$$\sum_{m=1}^{L} |S_m| \gg N^2. \tag{21.2}$$

Now with

$$A_m = \sum_{n=1}^{N} e(m(\alpha_1 n^2 + \beta_1 n)), \ B_m = \sum_{l=1}^{N} e(m(\alpha_2 l^2 + \beta_2 l))$$

we have

$$S_m = A_m B_m + A_m + B_m = A_m B_m + O(N) \ll |A_m|^2 + |B_m|^2 + O(N),$$

and therefore

$$\sum_{m=1}^{L} (|A_m|^2 + |B_m|^2) \gg N^2$$

by (21.2). We may suppose without loss of generality that

$$\sum_{m=1}^{L} |A_m|^2 \geq c_1 N^2,$$

where $c_1 = c_1(\epsilon) > 0$. We are going to apply Lemma 11A with $k = 2$, $\alpha = \alpha_1$, $\beta = \beta_1$, $A = c_1 N^2$, and with ϵ_1 in place of ϵ. The condition (11.2) is true for large N since

$$A^{1-\epsilon_1} \gg N^{2-2\epsilon_1} = N^{1+2\epsilon_1} N^{1-4\epsilon_1} \geq N^{1+2\epsilon_1} L.$$

There is a natural $q \leq LN^{2+\epsilon_1} A^{-1} \ll LN^{\epsilon_1} \ll N^{1-\epsilon_1}$ with

$$\|\alpha_1 q\| \leq N^{\epsilon_1} A^{-1} \ll N^{\epsilon_1 - 2}, \qquad \|\beta_1 q\| \leq N^{1+\epsilon_1} A^{-1} \ll N^{\epsilon_1 - 1}.$$

For sufficiently large N we obtain $q \leq N$ and $\|\alpha_1 q^2 + \beta_1 q\| \ll N^{\epsilon_1 - 1}$, and therefore $\|\alpha_1 q^2 + \beta_1 q\| \leq N^{\epsilon - 1}$. Thus (20.2) is true with $n = q$, $l = 0$.

We now turn to Theorem 20B. We set $I = N^{-c(s)+\epsilon}$ and we let $\mathfrak{I}$ be the interval $0 \leq x < I$. We apply Lemma 3A with $r > 1 + 3s^3\epsilon^{-1}$ to obtain a function $\psi(x)$. If the inequalities (20.4), (20.5) have no solution, then

$$\sum_{(n_1,\ldots,n_s)\in\mathfrak{N}} \psi\left(\sum_{i=1}^{s} (\alpha_i n_i^2 + \beta_i n_i)\right) = 0,$$

where $\mathfrak{N}$ is the set (20.4). We may infer that

$$\sum_{m\neq 0} c_m S_m \gg IN^s, \tag{21.3}$$

where

$$S_m = \sum_{(n_1,\ldots,n_s)\in\mathfrak{N}} e\left(m\left(\sum_{i=1}^{s} (\alpha_i n_i^2 + \beta_i n_i)\right)\right).$$

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With $\epsilon_1 = \epsilon/(3s^2)$ and $L = [I^{-1}N^{\epsilon_1}]$ we find that

$$\sum_{|m|\geq L} |c_m S_m| \ll N^s \sum_{|m|\geq L} I(I/|m|)^{-r} \ll N^s(I/L)^{1-r} \ll 1 = o(IN^s).$$

We thus obtain

$$\sum_{m=1}^{L} |S_m| \gg N^s \tag{21.4}$$

as a consequence of (21.3). The summands where $|S_m|$ is small compared to N^sL^{-1} give a small contribution to the sum. Hence there is a B with $N^sL^{-1} \ll B \leq N^s$ such that the set $\mathfrak{B}$ of integers $1 \leq m \leq L$ with

$$B \leq |S_m| < 2B \tag{21.5}$$

has $\Sigma_{m\in\mathfrak{B}}|S_m| \gg N^s/\log N$, or

$$|\mathfrak{B}| \gg N^s/(B\log N).$$

Putting

$$S_{mi} = \sum_{n=1}^{N} e(m(\alpha_i n^2 + \beta_i n))$$

we have

$$|S_m| \leq \prod_{i=1}^{s} (|S_{mi}| + 1). \tag{21.6}$$

Without loss of generality we may suppose that the subset $\mathfrak{B}'$ of $\mathfrak{B}$ consisting of m with $|S_{m1}| \geq \cdots \geq |S_{ms}|$, has cardinality

$$|\mathfrak{B}'| \gg N^s/(B\log N). \tag{21.7}$$

Again let h be the largest integer with $2h(h+1) \leq s$. We claim that for $m \in \mathfrak{B}'$,

$$|S_{mi}|^{2(1-\epsilon_1)} \geq N^{1+\epsilon_1} \quad (i = 1, \ldots, h). \tag{21.8}$$

For otherwise, we had $|S_{mi}| \leq N$, and $|S_{mi}| \leq N^{(1+\epsilon_1)/(2(1-\epsilon_1))} \leq N^{1/2+2\epsilon_1}$ for $h \leq i \leq s$, whence by (21.6),

$$|S_m| \ll N^{h-1+(1/2+2\epsilon_1)(s-h+1)} \leq N^{(1/2)(s+h-1)+2s\epsilon_1}$$

But it is easily seen that $5h \leq s+1$, whence $\frac{1}{2}(s+h-1) \leq s - 2h = s - c(s)$, so that

$$|S_m| \ll N^{s-c(s)+2s\epsilon_1} = o(N^{s-c(s)+\epsilon-\epsilon_1}) \ll N^sL^{-1} \ll B,$$

in contradiction to (21.5).

We now apply Lemma 11A with $k = 2$, $L = 1$, $\alpha = m\alpha_i$, $\beta = m\beta_i$, $A = |S_{mi}|^2$, and with ϵ_1 in place of ϵ. The condition (11.2) holds by (21.8). Accordingly, there is a natural r_i with

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(21.9)
$$r_i \le N^{2+\epsilon_1}|S_{mi}|^{-2},$$

(21.10)
$$\|\alpha_i m r_i\| < N^{\epsilon_1}|S_{mi}|^{-2}.$$

(The assertion of Lemma 11A about $\|\beta_i m r_i\|$ will not be used.) Such an integer r_i exists for $1 \le i \le h$ and for $m \in \mathfrak{B}'$. Now for $m \in \mathfrak{B}'$ we have $|S_{m1}| \cdots |S_{mh}| \gg |S_m|^{h/s} \gg B^{h/s}$. Thus if we write $q = r_1 r_2 \cdots r_h$, we get

$$q \le N^{2h+s\epsilon_1}B^{-2h/s}, \qquad \|\alpha_i m q\| < N^{2(h-1)+s\epsilon_1}B^{-2h/s} \quad (i = 1, \dots, h).$$

Such an integer $q = q(m)$ exists for every $m \in \mathfrak{B}'$. The product mq is $\le LN^{2h+s\epsilon_1}B^{-2h/s}$. Since the number of divisors m' of mq is $\ll N^{\epsilon_1}$, we obtain $\gg |\mathfrak{B}'|N^{-\epsilon_1} \gg N^{s-2\epsilon_1}B^{-1}$ distinct products mq as m runs through $\mathfrak{B}'$. There will be two such products whose difference is a natural number

$$z \ll (N^{s-2\epsilon_1}B^{-1})^{-1}LN^{2h+s\epsilon_1}B^{-2h/s}.$$

This number z will have

$$\|\alpha_i z\| \ll N^{2(h-1)+s\epsilon_1}B^{-2h/s} \quad (i = 1, \dots, h).$$

Thus

$$z\prod_{i=1}^{s}\|\alpha_i z\| \le z\prod_{i=1}^{h}\|\alpha_i z\| \ll LN^{2h^2-s+2s^2\epsilon_1}B^{1-(2h(h+1)/s)}.$$

Since the exponent of B here is nonnegative, and since $B \ll N^s$, we further obtain

$$z\prod_{i=1}^{s}\|\alpha_i z\| \ll LN^{-2h+2s^2\epsilon_1} \ll N^{2h-\epsilon+\epsilon_1}N^{-2h+2s^2\epsilon_1} \ll N^{-\epsilon_1}.$$

But since z is bounded by a certain power N^{c_2} of N, this is impossible for large N in view of the condition that $\alpha_1, \dots, \alpha_s$ be not very well approximable.

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