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Factorizations in Local Subgroups of Finite Groups

G. Glauberman

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To Gertrude and William Zwillinger

Preface

The past two decades have seen an extraordinary flowering of finite group theory. Most of this work has been aimed toward the specific problem of determining the finite simple groups. However, this pursuit has stimulated the study of other aspects of finite groups which are of interest in themselves as well as in their applications to simple groups. The purpose of this monograph is to describe some recent progress in one such aspect, that of Sylow subgroups. In particular, we address ourselves to the following question:

Given a prime p and a Sylow p-subgroup S of a finite group G, how is the structure of G influenced by the structure of S and the manner in which S is embedded in G?

We will focus on two related special cases of this question:

Question 1. *Which elements of S are conjugate in G?*

Question 2. *What is the relation between S and G if* $C(O_p(G)) \subseteq O_p(G)$?

The first important connections between S and G to be proved were obtained by Burnside and Frobenius about the turn of the century. However, their results applied only to special situations. The true strength of the connections between S and G in the general situation was first suggested by the techniques of John Thompson's Ph. D. dissertation in 1959. These techniques were developed and applied in the Odd Order Paper, Thompson's N-group paper, and many later articles. At a conference in Oxford in 1969, I gave a report on this subject entitled *Global and Local Properties of Finite Groups* (abbreviated here by GL); the proceedings of the conference appeared in the book, *Finite Simple Groups*, edited by M. B. Powell and G. Higman. The present monograph is based on lectures given at a conference in Duluth in 1976 and is intended to supplement GL by describing some of the progress since that time.

The text of GL was divided into two parts. The first half consisted of results by Alperin, Gorenstein, Thompson, and others, which developed a theoretical framework for investigating Question 1 and reducing it to Question 2. In the second half of GL, the methods of Thompson, Wielandt, and others were used to prove some special cases of Question 2 for odd p and thus obtain partial answers to Question 1.

The main part of this book follows the same pattern as GL, but with a different emphasis. The major goal in GL was to derive as much information as possible from the normalizer $N(W)$ of some characteristic subgroup W of S; this was done by reducing to special cases of Question 2 and then showing that $N(W) = G$. In contrast, the major goal in this book is to derive information from the normalizers of a *set* of characteristic subgroups by proving factorizations of the form $G = N(W_1)N(W_2)$. The reason for this is that recent results cover different territory (e.g., $p = 2$) from that in GL, and it has been possible (so far) to prove only factorizations, but not normality, for these results.

The body of this book is divided into three chapters as follows:

Chapter I. Here we prove some new results which extend the reductions of Question 1 in GL to cover an arbitrary set of characteristic subgroups rather than a single characteristic subgroup. To do this, we first review some concepts and results from the first half of GL. We end the chapter in a quite different vein with some important new work of Yoshida on transfer.

Chapter II. The second half of GL examined Question 2 in detail for p odd. In particular, one result (similar to the author's "ZJ-Theorem") gave a sufficient condition to have $N(W) = G$ for a specific nonidentity characteristic subgroup W of S. Unfortunately, the proof collapses completely for $p = 2$. Recently, a weak analogue for $p = 2$ has been obtained by using factorizations, and most of Chapter II is devoted to proving this result. In the last section of Chapter II, we apply this result and an important theorem of Goldschmidt to classify the simple groups in which the symmetric group of degree four is not involved.

Chapter III. Here we remove most of our earlier restrictions on G; in particular, we allow $SL(2, p)$ to be involved in G. In this situation, practically none of our previous machinery is applicable. We discuss without proofs a variety of new results and conclude with some applications, results in related areas, and open questions.

Having described what is contained in this book, we must describe what has been omitted. Fortunately, very little in the way of proofs. The topic of Sylow subgroups studied here differs from some other topics in finite group theory in not requiring an extensive background and long proofs. Indeed, it was only with extreme reluctance that we refrained from proving two theorems of Sylow at the beginning of GL. Except for these theorems and a few other quoted results, the main part of GL is self-contained. The present work assumes somewhat greater familiarity with elementary group theory but is otherwise largely self-contained (except for Chapter III, as mentioned above). There are three major exceptions.

(1) In Chapter I, many results are quoted from GL. (Since most of the results and techniques of the present work complement rather than generalize those of GL, we recommend but do not assume familiarity with GL.)

(2) Most of Chapter II is devoted to the proof of a single result (Theorem B). This result was first proved only for solvable groups. However, for the sake of applications to simple groups, the result was extended to allow composition factors of the form $PSL(2, 3^{2n+1})$ and $Sz(2^{2n+1})$. As we are interested only in using the latter groups rather than studying them, the reader unfamiliar with them is welcome merely to assume all the information about them needed in the proof of Theorem B (§§1–5) or simply to assume throughout that G is solvable. The same remarks apply to a preliminary result, Theorem A, which is proved in Appendix A1.

(3) The applications to simple groups and other topics at the end of Chapter II require several important facts which must be quoted without proof.

Unfortunately, there are many topics closely related to ours which space does not permit us to include. Our initial question asks how S influences the structure of G. In this book, we restrict our attention mainly to the influence of S on the internal, or 'local' structure of G, as exemplified by Questions 1 and 2. However, there is much research directed toward the 'global' structure of G, i.e., the structure of G as a whole. In particular, there are many results which yield the precise structure of G when $p = 2$, G is simple, and S (or the centralizer of an element of order two in S) is isomorphic to a given group. Unfortunately, most work of this type depends on such 'global' tools as ordinary character theory, block theory, or generators and relations. In comparison with the 'local' theory, papers in this area generally require more breadth, more depth, and more length. For this reason, we give only a brief sample of this work in the section on simple groups in Chapter II and in the quoted results in Chapter III.

Another large, related question that we have been forced to omit is how the structure of G is influenced by restrictions on its Sylow subgroups for more than one prime. In other words, how do the p-subgroups and q-subgroups of G interact when $q \neq p$, and how can we integrate information obtained for individual primes? Answers to this question have produced many applications to signalizer functors, uniqueness theorems for simple groups, and fixed-point-free groups of automorphisms. Fortunately, the reader can obtain an excellent introduction to this subject from three short papers [1], [11], [12] recommended as preliminary reading for participants in the conference. In particular, [1] gives an expository account of the entire area of relations between local and global information.

We emphasize one important convention which we will observe. Throughout the book, G will denote a fixed but arbitrary finite group; p, a fixed but arbitrary prime; and S, a fixed but arbitrary Sylow p-subgroup of G. Of course, all of the groups considered in this book are finite.

It is a great pleasure for me to thank the institutions which made the Duluth Conference and this book possible—the Conference Board of the Mathematical Sciences, the University of Minnesota at Duluth, and the National Science Foundation (which also assisted by research grants). I have benefited very much from the comments of the participants, especially I. M. Isaacs, who also contributed the material in §1.6 of this book. In addition, I am indebted to R. Niles and T. Yoshida for permission to use their unpublished work, and to Chat-Yin Ho for many corrections to GL which appear in the Appendix. Most of all, I wish to thank Joseph Gallian for a year of tireless effort to arrange, organize, and run the Conference superlatively.

I would also like to acknowledge my gratitude to L. A. Shepp and George Bachman, who began my education in group theory, and to J. G. Thompson and H. Wielandt for their further encouragement and stimulation. I am particularly grateful to R. H. Bruck as a teacher and friend for many years. In a sense, Daniel Gorenstein and Richard Lyons are the real authors of this book, because nearly all of its contents have grown out of two problems which they proposed two years ago.

Finally, I wish to express my thanks to Mr. Fred Flowers for an excellent job of typing under extremely trying circumstances and to my wife (and copy editor) and my children, Daniel and Rachel, for coping with similar circumstances.

University of Chicago George Glauberman

Preface to Second Printing

As mentioned above, this work supplements the author's earlier report, GL, on the connection between global and local properties of finite groups. Since the first printing, there has been significant progress in this area, and we have made corresponding additions to the text:

(1) Bernd Stellmacher has solved an outstanding open problem (Question 4.1 in Section 3.4) concerning an analogue to $\mathbf{K}_\infty$ and $\mathbf{ZJ}$ for $p = 2$. This yields a simpler proof of the most important result of Chapter II, the classification of non-abelian simple groups in which the symmetric group of degree four is not involved (Theorem C). His proof uses Section II.2 and Appendix A1 of this work. We have added a new section, Appendix A3, which is an account by the author of Stellmacher's article, written for Mathematical Reviews, with the kind permission of the American Mathematical Society.

(2) Several authors have solved other open questions in GL and Section 3.4. We have extended Section 3.4 to mention these solutions.

(3) There has been an explosion of activity in the area of Sections 3.1 and 3.2, "pushing-up" (including "failure of factorization"), especially after a landmark article by David Goldschmidt on amalgams in 1980. Space permits us only to mention here an expository introduction to new ideas in this subject in Sections 4.12 and 4.13 of Daniel Gorenstein's book [73], published in 1982.

In addition, all known errors have been corrected.

I thank Saunders Mac Lane for valuable advice on preparing this printing and Andrew Chermak and Ronald Solomon for information about current research.

University of Chicago **George Glauberman**

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