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MASAYOSHI NAGATA

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Thus if G is geometrically reductive, then R^G defines an affine variety which parameterizes closed orbits, or equivalently, the equivalence classes under $\sim$. But, in general, important part of orbits are usually orbits of biggest dimension. In that respect, Theorem 9.4 below may have some interest:

THEOREM 9.4. *Under the circumstances as above, assume furthermore that V is nonsingular. Then for each point p of V, $(R_p)^G$ is a ring of quotients of a finitely generated ring over K.*

PROOF. We admit the fact that if D is a divisorial closed subset of a nonsingular affine variety V, then $V - D$ is an affine variety (see, for instance, Nagata [**10**]). If $(R_p)^G = (R^G)_{m_p \cap R^G}$, then the assertion is obvious. Assume the other case. Then there is a G-invariant rational function f in R_p which is not in $(R^G)_{m_p \cap R^G}$. Let D be the pole of f and we reduce V to $V - D$. This may be repeated, but, since $F(p)$ contains only one closed orbit and since $\dim F(p)$ is finite, the reduction terminates at a finite step. Thus we obtain the conclusion. Q. E. D.

When we observe a rational action of an algebraic group G on an affine, or projective, or more generally, abstract variety or scheme, we often pay attention to some "good part" U. By some reasons, an open set U satisfying the following condition (**) may be regarded as a kind of good part (cf. Mumford [**9**]):

(**) $$\{(p, \sigma p) \mid p \in U, \sigma \in G\} \text{ is a closed subset in } U \times U.$$

Another remark we want to give here is that there is an example in case G is SL$(3, K)$ and V is the affine space of dimension 21 such that U satisfies (**) but U is not very nice. For the detail of the example, the readers are advised to see Nagata [**16**].

In connection with this Theorem 9.4, we note here that if we drop the condition that V is nonsingular and assume only that V is normal, then the conclusion becomes false; an example was given by Nagata [**16**].

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REFERENCES

1. J. W. Alexander, *On the factorization of Cremona plane transformations,* Trans. Amer. Math., Soc. **17** (1916), 295–300.

2. G. Castelnuovo, *Recerche generali sopra i systemi lineari di curve piani,* Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. ser. 2, **42** (1892), 3–43.

3. ———, *La trasformazioni generatrici del gruppo Cremoniano nel piano,* Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **36** (1901) 861–874.

4. W. Haboush, *Reductive groups are geometrically reductive,* Ann. of Math. (2) **102** (1975), 67–83.

5. H. P. Hudson, *Cremona transformations,* Cambridge Univ. Press, 1927.

6. H.W.E. Jung, *Zusammensetzung von Cremonatransformationen der Ebene aus quadratischen Transformationen,* Crelle J. **180** (1939), 97–109.

7. ———, *Über ganze birationale Transformationen der Ebene,* Crelle J. **184** (1942), 161–174.

8. W. van der Kulk, *On polynomial rings in two variables,* Nieuw Arch. Wisk. (3) **1** (1953), 33–41.

9. D. Mumford, *Geometric invariant theory,* Springer, Berlin, 1965.

10. M. Nagata, *A treatise on the* 14*th problem of Hilbert,* Mem. Coll. Sci. Univ. Kyoto, A-Math. **30** (1956–57), 57–70; addition and correction, ibid., 197–200.

11. ———, *On the* 14*th problem of Hilbert,* Amer. J. Math. **81** (1959), 766–772.

12. ———, *On the* 14*th problem of Hilbert,* Sûgaku **12** (1960), 203–209. (Japanese)

13. ———, *On rational surfaces.* I, Mem. Coll. Sci. Univ. Kyoto, A-Math. **32** (1959–60), 351–370.

14. ———, *On rational surfaces.* II, Mem. Coll. Sci. Univ. Kyoto, A-Math. **33** (1960–61), 271–293.

15. ———, *Complete reducibility of rational representations of a matric group,* J. Math. Kyoto Univ. **1** (1961–62), 87–99.

16. ———, *Note on orbit spaces,* Osaka Math. J. **14** (1962), 21–31.

17. ———, *Invariants of a group in an affine ring,* J. Math. Kyoto Univ. **3** (1963–64), 369–377.

18. ———, *Lectures on the fourteenth problem of Hilbert,* Tata Inst. F. R., 1965.

19. ———, *On automorphisms group of* $k[x, y]$, Lectures Math. Kyoto Univ., Kon-okuniya, Tokyo, 1972.

20. D. Rees, *On a problem of Zariski,* Illinois J. Math. **2** (1958), 145–149.

21. B. L. van der Waerden, *Einfürung in die algebraische Geometrie,* Springer, Berlin, 1939.

http://dx.doi.org/10.1090/cbms/037/10

22. R. Weitzenböck, *Über die Invarianten von linear Gruppen,* Acta Math. **58** (1932), 231–293.

23. H. Weyl, *The classical groups,* Princeton Univ. Press, Princeton, N.J., 1939.

24. O. Zariski, *Foundations of general theory of birational correspondences,* Trans. Amer. Math. Soc. **53** (1943), 490–542.

25. ———, *Interprétations algébrico-géométriques du quatorzième problème de Hilbert,* Bull. Sci. Math. **78** (1954), 155–168.

26. ———, *Introduction to the problem of minimal models in the theory of algebraic surfaces,* Publ. Math. Soc. Japan **4** (1958).

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