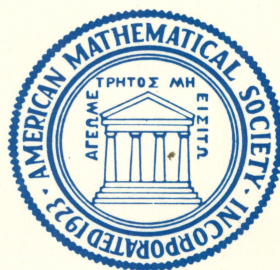


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number 37



**MASAYOSHI NAGATA**

**POLYNOMIAL RINGS  
AND AFFINE SPACES**



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published by the american mathematical society

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MASAYOSHI NAGATA

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Thus if  $G$  is geometrically reductive, then  $R^G$  defines an affine variety which parameterizes closed orbits, or equivalently, the equivalence classes under  $\sim$ . But, in general, important part of orbits are usually orbits of biggest dimension. In that respect, Theorem 9.4 below may have some interest:

**THEOREM 9.4.** *Under the circumstances as above, assume furthermore that  $V$  is nonsingular. Then for each point  $p$  of  $V$ ,  $(R_p)^G$  is a ring of quotients of a finitely generated ring over  $K$ .*

**PROOF.** We admit the fact that if  $D$  is a divisorial closed subset of a nonsingular affine variety  $V$ , then  $V - D$  is an affine variety (see, for instance, Nagata [10]). If  $(R_p)^G = (R^G)_{m_p \cap R^G}$ , then the assertion is obvious. Assume the other case. Then there is a  $G$ -invariant rational function  $f$  in  $R_p$  which is not in  $(R^G)_{m_p \cap R^G}$ . Let  $D$  be the pole of  $f$  and we reduce  $V$  to  $V - D$ . This may be repeated, but, since  $F(p)$  contains only one closed orbit and since  $\dim F(p)$  is finite, the reduction terminates at a finite step. Thus we obtain the conclusion. Q.E.D.

When we observe a rational action of an algebraic group  $G$  on an affine, or projective, or more generally, abstract variety or scheme, we often pay attention to some “good part”  $U$ . By some reasons, an open set  $U$  satisfying the following condition (\*\*) may be regarded as a kind of good part (cf. Mumford [9]):

$$(**) \quad \{(p, \sigma p) \mid p \in U, \sigma \in G\} \text{ is a closed subset in } U \times U.$$

Another remark we want to give here is that there is an example in case  $G$  is  $SL(3, K)$  and  $V$  is the affine space of dimension 21 such that  $U$  satisfies (\*\*) but  $U$  is not very nice. For the detail of the example, the readers are advised to see Nagata [16].

In connection with this Theorem 9.4, we note here that if we drop the condition that  $V$  is nonsingular and assume only that  $V$  is normal, then the conclusion becomes false; an example was given by Nagata [16].



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