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Edward G. Effros

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Contents

To Rita, Rachel and Stephen

Preface

These notes parallel a series of ten lectures that I gave at the Conference Board of the Mathematical Sciences Regional Conference in Rochester, Michigan, June 1979, sponsored by the National Science Foundation and the Department of Mathematical Sciences, Oakland University. I have taken the liberty of including some new results as well as some improved arguments suggested by colleagues at the conference.

I am most grateful to Jack Tsui, Steve Wright, and the CBMS for organizing the conference and to the National Science Foundation for its support. The facilities of Meadow Brook Hall were elegant and conducive to the fruitful interaction of the participants. I am also indebted to Alain Connes, who provided me with an opportunity to prepare these lectures in a visit at the Institut des Hautes Études Scientifiques. During the preceding year I had profited from a number of conversations with him regarding his approach to geometry, some aspects of which are reflected in Chapter 1. I also wish to thank the participants, who provided an audience that was both enthusiastic and provocatively skeptical, as well as my colleagues who attended a course at UCLA in which I had the opportunity to expand the lecture notes.

Finally I wish to thank my collaborator and former student Chao-Liang Shen. His unflagging enthusiasm and ingenuity have provided considerable impetus to the further development of dimension groups as a new mathematical discipline.

August 1980
UCLA

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References

1. E. M. Alfsen, *Compact convex sets and boundary integrals*, Ergebnisse der Math., vol. 57, Springer-Verlag, Berlin, 1971.

2. V. I. Arnold and A. Avez, *Ergodic problems of classical mechanics*, Mathematical Physics Monograph Series, Benjamin, New York, 1968.

3. M. G. Atiyah, *K-theory*, Benjamin, New York, 1967.

4. B. Blackadar, *Traces on simple AF algebras*, J. Functional Anal. **38** (1980), 156–168.

5. ———, *A simple unital projectionless C^*-algebra*, J. Operator Theory (to appear).

6. H. Bohr, *Ueber fastperiodische ebene Bewegungen*, Comment. Math. Helv. **4** (1934), 51–64.

7. Z. Borevich and I. Shefarevich, *Number theory*, Academic Press, New York, 1966.

8. O. Bratteli, *Inductive limits of finite dimensional C^*-algebras*, Trans. Amer. Math. Soc. **171** (1972), 195–234.

9. O. Bratteli and G. Elliott, *Structure spaces of approximately finite dimensional C^*-algebras*. II, J. Functional Anal. **30** (1978), 74–82.

10. O. Bratteli, G. Elliott and R. Herman, *On the possible temperatures of a dynamical system*, Comm. Math. Phys. **74** (1980), 281–295.

11. J. Bunce and J. Deddens, *A family of simple C^*-algebras related to weighted shift operators*, J. Funct. Anal. **19** (1975), 13–24.

12. G. Choquet, *Lectures on analysis*, Benjamin, New York, 1969.

13. L. Coburn, R. Douglas, D. Schaeffer and I. M. Singer, *C^*-algebras of operators on a half-space*. II, *Index theory*, Inst. Hautes Études Sci. Publ. Math. **40** (1971), 69–79.

14. A. Connes, *Classification of injective factors*, Ann. of Math. **104** (1976), 73–115.

15. ———, *Sur la théorie non-commutative de l'integration*, Lecture Notes in Math., vol. 725, Springer-Verlag, Berlin and New York, 1979, pp. 19–143.

16. ———, *An analogue of the Thom isomorphism for cross products of a C^*-algebra by an action of* **R**, Adv. in Math. (to appear).

17. ———, *C^*-algèbres et géométrie differentielle*, C. R. Acad. Sc. Paris **290** (1980), 599–604.

18. J. Cuntz, *K-theory for certain C^*-algebras*, Ann. of Math. **113** (1980), 181–197.

19. ———, *K-theory for certain C^*-algebras*. II, J. Operator Theory (to appear).

20. J. Cuntz and W. Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math. **56** (1980), 251–268.

21. ———, *Topological Markov chains with dicyclic dimension groups,* Crelles J. (to appear).

22. M. Denker, C. Grillenberger and K. Sigmund, *Ergodic theory on compact spaces,* Lecture Notes in Math., vol. 527, Springer-Verlag, Berlin, 1976.

23. J. Dixmier, *Les C^*-algèbras et leurs représentations,* Gauthier-Villars, Paris, 1964.

24. ———, *On some C^*-algebras considered by Glimm,* J. Funct. Anal. **1** (1967), 182–203.

25. E. Effros, *Order ideals in a C^*-algebra and its dual,* Duke Math. J. **30** (1963), 391–412.

26. E. Effros and F. Hahn, *Locally compact transformation groups and C^*-algebras,* Mem. Amer. Math. Soc., no. 75, Amer. Math. Soc., Providence, R. I., 1967.

27. E. Effros, D. Handelman and C.-L. Shen, *Dimension groups and their affine representations,* Amer. J. Math. **102** (1980), 385–407.

28. E. Effros and J. Rosenberg, *C^*-algebras with approximately inner flip,* Pacific J. Math. **77** (1978), 417–443.

29. E. Effros and C.-L. Shen, *Approximately finite C^*-algebras and continued fractions,* Indiana J. Math. **29** (1980), 191–204.

30. ———, *The geometry of finite rank dimension groups,* Illinois J. Math. **25** (1981), 27–38.

31. ———, *Dimension groups and finite difference equations,* J. Operator Theory **2** (1980), 215–231.

32. G. Elliott, *Automorphisms determined by multipliers on ideals of a C^*-algebra,* J. Funct. Anal. **23** (1976), 1–10.

33. ———, *On the classification of inductive limits of sequences of semi-simple finite dimensional algebras,* J. Algebra **38** (1976), 29–44.

34. ———, *On totally ordered groups and K_0*, Ring Theory Conference (Waterloo, 1978), Lecture Notes in Math., vol. 734, Springer-Verlag, Berlin and New York, 1979.

35. Th. Fack and O. Maréchal, *Application de la K-théorie algèbrique aux C^*-algèbres,* Lecture Notes in Math., vol. 725, Springer-Verlag, Berlin and New York, 1979, pp. 144–169.

36. L. Fuchs, *Riesz groups,* Ann. Scuola Norm. Sup. Pisa **19** (1965), 1–34.

37. F. Gantmacher, *Applications of the theory of matrices,* Interscience, New York, 1959.

38. J. Glimm, *On a certain class of operator algebras,* Trans. Amer. Math. Soc. **95** (1960), 318–340.

39. K. Goodearl and D. Handelman, *Rank functions and K_0 of regular rings,* J. Pure Appl. Algebra **7** (1976), 195–216.

40. ———, *Metric completions of partially ordered abelian groups,* Indiana Univ. Math. J. **29** (1980), 861–895.

41. D. Handelman, *K_0 of von Neumann algebras and AF C^*-algebras,* Quart. J. Math. Oxford (2) **29** (1978), 427–441.

42. ———, *Positive integral matrices and totally ordered groups*, J. Operator Theory (to appear).

43. H. Helson, *Analyticity on compact abelian groups*, Algebras in Analysis (J. H. Williamson, ed.), Academic Press, New York, 1975.

44. R. Herman and V. F. R. Jones, *Period two automorphisms of UHF algebras* (to appear).

45. E. Hewitt and K. Stromberg, *Real and abstract analysis*, Springer-Verlag, New York, 1965.

46. E. Hille and R. Phillips, *Functional analysis and semigroups*, Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc., Providence, R. I., 1957.

47. R. Kadison, *A representation theory for commutative topological algebra*, Mem. Amer. Math. Soc., no. 7, Amer. Math. Soc., Providence, R. I., 1951.

48. W. Krieger, *On dimension functions and topological Markov chains*, Invent. Math. **56** (1980), 239–250.

49. A. Kumjian, *On localizations and simple C^*-algebras*, Thesis, University of California, Berkeley, 1980.

50. F. Murray and J. von Neumann, *On rings of operators*, Ann. of Math. **37** (1936), 116–229.

51. G. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, New York, 1979.

52. M. Pimsner and D. Voiculescu, *Imbedding the irrational rotation algebra into an AF algebra*, J. Operator Theory **4** (1980), 201–210.

53. ———, *Exact sequences for K groups and* Ext *groups of certain crossed product C^*-algebras*, J. Operator Theory **4** (1980), 93–118.

54. N. Riedel, *Classification of dimension groups and iterating systems*, Math. Scand. (to appear).

55. J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Math., vol. 793, Springer-Verlag, Berlin and New York, 1980.

56. M. Rieffel, *Irrational rotation algebras*, Proc. Internat. Congr. Math. (Helsinki, 1978), Amer. Math. Soc., Providence, R. I., 1979, short communication.

57. C.-L. Shen, *On the classification of the ordered groups associated with the approximately finite dimensional C^*-algebras*, Duke Math. J. **46** (1979), 613–633.

58. J. L. Taylor, *Banach algebras and topology*, Algebras in Analysis (J. H. Williamson, ed.), Academic Press, New York, 1975.

59. P. Walters, *Ergodic theory–introductory lectures*, Lecture Notes in Math., vol. 458, Springer-Verlag, Berlin and New York, 1975.

60. R. F. Williams, *Classification of one-dimensional attractors*, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R. I., 1970, pp. 341–361.

61. ———, *Classification of subshifts of finite type*, Ann. of Math. **98** (1973), 120–153; Errata **99** (1974), 380–381.

62. Ando, *On fundamental properties of a Banach space with a cone*, Pacific J. Math. **12** (1962), 1163–1169.

63. D. Handelman, *An algorithmic approach to ultrasimplicial simple dimension groups* (to appear).

64. J. Lindenstrauss, *Extension of compact operators,* Mem. Amer. Math. Soc., no. 48, Amer. Math. Soc., Providence, R. I., 1964.

65. S. Eilenberg and N. Steenrod, *Foundations of algebraic topology,* Princeton Univ. Press, Princeton, N. J., 1952.

66. L. Brown, *Extensions of AF algebras: the projection lifting problem,* (Proc. 1980 Summer Research Inst. on Oper. Alg. and Appl., Kingston) (to appear).

67. E. B. Davies and G. Vincent-Smith, *Tensor products, infinite products, and projective limits of Choquet simplexes,* Math. Scand. **22** (1968), 145–164.

68. W. Parry and R. F. Williams, *Block coding and zeta function for finite Markov chains,* Proc. London Math. Soc. **35** (1977), 483–495.

69. L. Asimow and A. Ellis, *Convexity and its applications in functional analysis,* Academic Press, New York, 1980.

70. A. Connes, *A survey of foliations and operator algebras,* I. H. E. S. Preprint, 1981.

71. A. Phillips and D. Sullivan, *Geometry of leaves,* Topology **20** (20), 209–218.

72. N. Riedel, *A counterexample to the unimodular conjecture on finitely generated dimension groups,* preprint, 1981.

73. S. Schwartzman, *Asymptotic cycles,* Ann. of Math. **66** (1957), 270–284.

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More recent developments in dimension groups

A supplement to *Dimensions and C^*-algebras* by Edward Effros

by David Handelman

In the almost fifteen years since the expository lecture notes on *Dimensions and C^*-Algebras* were written by Ed Effros, there have been a large number of exciting developments. Many of these are still in flux, and so it is impossible (or at least very frustrating for potential readers) to give references to all the relevant work; much of it is circulating in preprint form. Instead, I shall give the names of some of the people who are active in these areas, and hope that those interested will contact them directly. The order of discussion of the topics (which overlap somewhat) is not intended to be either chronological or in relative importance.

Of course, a very important and active topic relevant to dimensions and C^*-algebras is the study of inclusions of von Neumann algebras of finite index, knot theory, the Jones polynomial and subsequent multivariable versions, etc. This is now so vast an area that it would be impossible to do it justice in a brief discussion such as this, nor am I qualified to present any sort of summary of it.

For a systematic development of dimension groups and their relatives, the book [G] is a very useful reference.

Classification of simple C^*-algebras. In section 10, a conjecture is made about the structure of irrational rotation C^*-algebras, namely that they can be realized as a C^*-direct limit of direct sums of matrix algebras over $C(\mathbf{T}^2)$. In fact a much better result was established, the culminating theorem (based on earlier work by Riedel, Choi and Elliott, and others) being that irrational rotation algebras are direct limits over $C(\mathbf{T})$ (the actual result being better than even this), by Elliott and Evans. This is part of a much larger programme, formulated by George Elliott, to classify simple C^*-algebras.

The idea is that K-theoretic data might be enough to classify C^*-algebras that are "sufficiently non-commutative", and in particular certain types of simple C^*-algebras. A particularly attractive class is that consisting of direct limits of matrix algebras over continuous functions on various compact Hausdorff spaces, or more generally, direct limits of reasonable type I C^*-algebras. There are at least two aspects to this problem. One is to find reasonable invariants; the second is to be able to decide if the C^*-algebra you have in front of you belongs to this class. The initial set of invariants consists of $K_0 \oplus K_1$ equipped with a partial ordering (which essentially makes K_1 into a group of infinitesimals). Since traces need not be determined by their effect on K_0, there is an additional datum that is not determined by the Grothendieck group, the natural pairing $\mathrm{T}((A)) \times K_0 \to \mathbb{R}$, where $\mathrm{T}(A)$ denotes the trace space of A. At the time of writing (according to George Elliott), these invariants are complete for certain classes of inductive limits of matrix algebras over spaces of finite dimension. However, there are very recent examples which show that these invariants do not suffice for certain related inductive limits. Remarkably, it has been proved that additonal K-theoretic invariants suffice to classify these more general limits. For a survey of this (and a list of references), see [Ell].

Dynamical systems, part I. Shifts of finite type (aka *topological Markov chains*) were discussed briefly in chapter 6. Williams discovered a relatively easily computed invariant, shift equivalence, and Krieger formalized the associated dimension group (constructed topologically) as part of a complete invariant for shift equivalence. Williams' conjecture, that shift equivalence implies strong shift equivalence, is still open in the most important case (primitive integer matrices), but is known to fail in many other situations—for example, reducible matrices (Kim, Rousch, and Wagoner), primitive matrices with nonnegative polynomial entries (Boyle), and others. However, it turned out that the dimension group construction was a very useful tool in this area. Let A and B be primitive $0-1$ matrices (this can be replaced by primitive integer matrices) of sizes $n(A)$, $n(B)$ respectively; let (X_A, T_A) etc., be their respective shift spaces, that is, $X_A = \{(x_i) \in \{1, 2, \ldots, n(A)\} \mid A(x_i, x_{i+1}) = 1\}$, and T_A is just the restriction of the bilateral shift to X_A, which is an invariant subset. The dimension group with autormophism attached to A, namely $(G_A, \hat{A})$ is defined in section 6. Sample theorem (one of many in [BMT, Theorem 3.2]):

> If there is an onto (order preserving) group homomorphism $G_A \to G_B$ that intertwines the order automorphisms $\hat{A}$ and $\hat{B}$, then for all sufficiently large m, there are one to one a.e. factor maps from (X_{A^m}, T_{A^m}) onto (X_{B^m}, T_{B^m}) (and these are given by right closing maps).

Dynamical systems, part II. Based on earlier work by Anatoly Vershik and Ian Putnam (separately), Herman, Putnam, and Skau ([HPS]) gave a miraculous description and classification of minimal homeomorphisms of metrizable boolean spaces. Given a Bratteli diagram for a simple dimension group, place an ordering on the path space (essentially) so that paths which agree except at finitely many levels are comparable, and conversely comparable paths are cofinal. A partially defined transformation on the path space, given by sending a path to its successor in its ordering may be extended to a homeomorphism on the path space if there are unique minimal and maximal paths. Amazingly, every minimal homeomorphism of a Cantor set is conjugate to one of these constructions. Moreover, they showed that two such systems are strongly orbit equivalent if and only if the dimension groups are (unitally) order isomorphic (two pairs of spaces with self-homeomorphisms (X, ϕ) and (Y, ψ) are *strongly orbit equivalent* if there exists a homeomorphism $\alpha\ X \to Y$ —orbits are sent onto orbits— this by itself is *orbit equivalence*—and the functions $n\ X \to \mathbf{Z}$ and $m\ Y \to \mathbf{Z}$ described by $\alpha\phi(x) = \psi^{m(x)}\alpha(x)$ and $\alpha^{-1}\psi(y) = \phi^{n(y)}\alpha^{-1}(y)$ are both continuous except possibly at one point). Earlier work by Putnam had shown that inside the crossed product are naturally occurring AF algebras, whose ordered Grothendieck group is that of the crossed product, and this agrees with the dynamicists first cohomology group—except that there was no ordering on the latter, whereas the ordering on the K_0 group is essential. This also lead to many of the classification results mentioned earlier in the section on simple C^*-algebras. This makes constructions of interesting examples relatively easy; e.g., every minimal homeomorphism on a Cantor set is (strongly) orbit equivalent to one of entropy zero, and the dyadic adding machine is (strongly) orbit equivalent to minimal homeomorphisms of any entropy (including infinite).

Very recently, Giordano, Putnam, and Skau showed that orbit equivalence for these minimal homeomorphisms is completely classified by the underlying dimensio group (K_0 of the crossed product, viewed as a unital ordered group) factored out by its subgroup of infinitesimals. Since orbit equivalences are usually rather nasty, this is a striking result.

There has also been some work on non-minimal zero dimensional dynamical systems from this point of view (Boyle and Handelman); for irreducible shifts of finite type, the K_0 of the crossed product (viewed as an ordered group) is a complete invariant for flow equivalence. As a consequence, in this class of dynamical systems, orbit equivalence implies flow equivalence, which is quite different from the minimal situation. (Still open is whether shift equivalence implies anything better than flow equivalence of all powers of the shifts.)

Group actions. Perhaps the most striking result here was the example of Blackadar answering in the negative one of the questions in 10.3. There exists an order two automorphism of 2^∞ UHF algebra whose fixed point algebra is not AF [Bla]. In fact, the order two automorphism can be defined on a dense subalgebra which is a direct limit of matrices over continuous functions on the circle. This and later generalizations motivated and blended naturally into the classification programme for simple C^*-algebras.

Fack and Marechal ([FM1], [FM2]) considered actions of product type (infinite tensor product actions) of prime order groups on UHF algebras, and showed (in effect) that these actions were classifiable by the K_0 of the crossed product (when viewed as a module over the dual group, acting via the dual action). This was extensively generalized by Handelman and Rossmann ([HR1], [HR2]) to compact (not necessarily abelian) groups of automorphisms acting on a dense nested union of finite dimensional algebras so that all the automorphisms are approximately inner (now the Grothendieck group becomes an ordered module over the representation ring of the group). Describing the ordering on the resulting dimension group now leads to various probabilistic questions (e.g., [H1, H2]). In another direction, classification of these locally definable automorphisms in the case of order 2 is a special case of the classification of real AF C^*-algebras (where we allow matrix rings over the reals, complexes, or quaternions to appear in any combination in the direct limit). This was due independently by Giordano [Gi], Goodearl and Handelman [GH], and Stacey [St].

Weak ergodicity. A sequence of nonnegative real matrices $\{A_i\}$ is *weakly ergodic* if for all k, the angles between the columns of the products $A_{N+k}A_{N+k-1}\dots A_k$ converges to zero as $N \to \infty$ [Se, chapter 3]. Let G be a simple dimension group given as a limit of strictly positive matrices $\lim A_i\ \mathbf{Z}^{n(i)} \to \mathbf{Z}^{n(i+1)}$. Then G will have a unique state if and only if the sequence $\{A_i\}$ is weakly ergodic. Not much is known about weak ergodicity of sequences of matrices, except for consequences of the Birkhoff contraction theorem (op. cit.), and it would be useful interesting to have some perturbation results. Let us alter the usual definition of dimension groups to have limits of finite dimensional real vector spaces (i.e., $\mathbb{R}^{n(i)}$ in place of $\mathbf{Z}^{n(i)}$). Then assuming ε_i are bounded nonnegative real numbers, the sequence defined by

$$A_i := \begin{bmatrix} 2 & \varepsilon_i \\ \varepsilon_i & 2 \end{bmatrix}$$

is weakly ergodic if and only if $\sum \varepsilon_i = \infty$ (this is easy to check—if $\{A_i\}$ is a family of commuting primitive matrices, weak ergodicity holds if and only if $\sum(1 - \delta(A_i)) = \infty$, where $\delta(A)$ is the ratio of the second largest absolute value of an eigenvalue to the spectral radius); evidently, the perturbation from $\left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right]$ has to be substantial in order to give a unique state. However, the sequence of matrices defined by

$$B_i := \begin{bmatrix} 2 & 1/2^i \\ 1/2^i & 1 \end{bmatrix}$$

is also weakly ergodic despite the fact that the perturbation from the diagonal matrix is now very small in most reasonable senses. Problems concerning harmonic functions on fairly general infinite state Markov chains can often be reduced to questions on relatively small dimension groups, and weak ergodicity results would represent a first step in attacking them.

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References for supplement

[Bla] Bruce Blackadar, *Symmetries of the CAR algebra*, Annals of Math. **131** 1990, 589–623.

[Ell] George A. Elliott, *The classification problem for amenable C*-algebras*, Proceedings ICM, Zurich 1994, (to appear).

[BMT] M. Michael Boyle, Brian Marcus, and Paul Trow, *Resolving maps and the dimension group for shifts of finite type*, Memoirs of the American Mathematical Society **377**, American Math. Soc., 1987.

[FM1] Thierry Fack & Odile Marechal, *Sur la classification des symétries des C*-algèbres UHF*, J. Canadien de Math. **31** 1979, 496–523.

[FM2] ———, *Sur la classification des automorphismes périodiques des C*-algèbres UHF*, J. Functional Analysis **40** 1981, 267–301.

[Gi] Thierry Giordano, *Classification of approximately finite dimensional real C*-algebras*, Crelle's Journal **385** 1988, 161–194.

[G] Kenneth R. Goodearl *Partially ordered abelian groups with interpolation*, Surveys and Monographs **20** 1986, American Mathematical Society.

[GH] Kenneth R. Goodearl & David Handelman, *Classification of ring and C*-algebra direct limits of fintie dimensional semisimple real algebras*, Memoirs of the American Math. Soc. **372** 1987, American Math. Soc.

[H1] David Handelman, *Deciding eventual positivity of polynomials*, Ergodic Theory and Dynamical Systems, **6** 1986, 57–79.

[H2] ——— *Representation rings as invariants fro compact groups and limit ratio theorems for them*, International J. of Math. **4** 1993, 59–93.

[HR1] David Handelman & Wulf Rossmann, *Product type actions of finite and compact groups*, Indiana J. of Math. **33** 1984, 479–509.

[HR2] ——— *Actions of compact groups on AF C*-algebras*, Illinois J. of Math. **29** 1985, 51–95.

[HPS] Richard Herman, Ian Putnam, & Christian Skau, *Ordered Bratteli diagrams, dimension groups, and topological dynamics*, International J. of Math. **3** 1992, 827–864.

[Se] Eugene Seneta, *Non-negative matrices and Markov chains*, Springer Series in Statistics, 1981, Springer Verlag.

[St] Peter J. Stacey, *Real structure in direct limits of finite dimensional C* algebras*, Journal of the London Math. Soc. **35** 1987, 339–352.

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