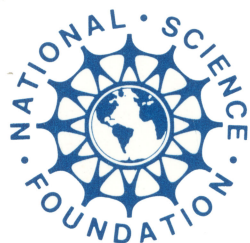


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number **50**



JAMES EELLS and LUC LEMAIRE
SELECTED TOPICS
IN HARMONIC MAPS



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by
JAMES EELLS AND LUC LEMAIRE

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