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Contents

Preface

These lectures are designed to provide a survey of modern intersection theory in algebraic geometry. This theory is the result of many mathematicians' work over many decades; the form espoused here was developed with R. MacPherson.

In the first two chapters a few epsisodes are selected from the long history of intersection theory which illustrate some of the ideas which will be of most concern to us here. The basic construction of intersection products and Chern classes is described in the following two chapters. The remaining chapters contain a sampling of applications and refinements, including theorems of Verdier, Lazarsfeld, Kempf, Laksov, Gillet, and others.

No attempt is made here to state theorems in their natural generality, to provide complete proofs, or to cite the literature carefully. We have tried to indicate the essential points of many of the arguments. Details may be found in [**16**].

I would like to thank R. Ephraim for organizing the conference, and C. Ferreira and the AMS staff for expert help with preparation of the manuscript.

Preface to the 1996 printing

In this revision, we have taken the opportunity to correct some errors and misprints. In addition, a section of notes has been added, to point out some of the work that has been done since the first edition was written that is closely related to ideas discussed in the text. Superscripts in the text refer to these notes. As in the text, no attempt is made to survey the large and growing literature in intersection theory.

I am grateful to Jeff Adler for preparing and improving the manuscript and diagrams.

William Fulton
Chicago, IL
December, 1995

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Notes (1983–1995)

1. (p. 5) For a description of the intersection ring of the space of complete quadrics, see

> C. De Concini and C. Procesi, *Complete symmetric varieties, II. Intersection theory*, in *Algebraic Groups and Related Topics*, Advanced Studies in Pure Math., vol. 6, North-Holland, 1985, pp. 481–513.

The Chow ring of these varieties is still only partially understood.

2. (pp. 20, 69) An elementary construction of this fundamental class, following [**23**], is given in Appendix B of

> W. Fulton, *Young tableaux, with applications to representation theory and geometry*, Cambridge University Press, to appear.

3. (p. 23) Grothendieck had proved this in [**10**] under the weaker assumption that E is an affine bundle over X. Gillet proved it with no group acting on the bundle, in

> H. Gillet, *Riemann-Roch theorems for higher algebraic K-theory*, Advances in Math. **40** (1981), 203–289.

For an application to a stronger splitting principle for Chow groups, see the second reference in Note 10.

4. (p. 33) For more along these lines, see

> R. Smith and R. Varley, *Singularity theory applied to Θ-divisors*, Springer Lecture Notes in Mathematics **1479** (1991), 238–257.

5. (p. 39) Another proof of this functoriality can be found in

> A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math. **97** (1989), 613–670.

6. (p. 41) Although computing Chow groups and rings of general smooth projective varieties remains a very hard problem, there are now many more varieties about which something is known. A careful survey of this could take a volume by itself. Here is a small sampling of references:

A. Beauville, *Sur l'anneau de Chow d'une variété abélienne*, Math. Annalen **273** (1986), 647–651.

G. Ellingsrud and S. A. Strømme, *On the Chow ring of a geometric quotient*, Ann. of Math. **130** (1989), 159–187.

A. Collino and W. Fulton, *Intersection rings of spaces of triangles*, Mém. Bull. Soc. Math. France **117** (1989), 75–117.

C. Faber, *Chow rings of moduli spaces of curves. I. The Chow ring of* $\overline{M}_3$; *II. Some results on the Chow ring of* $\overline{M}_4$, Ann. of Math. **132** (1990), 331–419, 421–449.

G. Ellingsrud and S. A. Strømme, *Towards the Chow ring of the Hilbert scheme of* $\mathbb{P}^2$, J. Reine Angew. Math. **441** (1993), 33–44.

S. Keel, *Intersection theory of moduli space of stable n-pointed curves of genus zero*, Trans. Amer. Math. Soc. **330** (1992), 545–574.

K. H. Paranjape, *Cohomological and cycle-theoretic connectivity*, Ann. of Math. **139** (1994), 641–660.

W. Fulton, R. MacPherson, F. Sottile, and B. Sturmfels, *Intersection theory on spherical varieties*, J. Alg. Geom. **4** (1995), 181–193.

Many other calculations of Chow groups are contained in other papers mentioned elsewhere in these notes.

7. (p. 44) For higher degrees, it is still the case that only a few of these numbers are known. For some modern work on this, see

P. Aluffi, *The enumerative geometry of plane cubics I: smooth cubics*, Trans. Amer. Math. Soc. **317** (1990), 501–539.

S. Kleiman and R. Speiser, *Enumerative geometry of nonsingular plane cubics*, in *Algebraic Geometry: Sundance 1988*, Contemp. Math. **116** (1991), 85–113.

8. (p. 50) These formulas are now special cases of a general formula for degeneracy loci of maps between two bundles with flags of subbundles. There is such a locus for each permutation, and the corresponding formula is given by the corresponding "double Schubert polynomial" of Lascoux and Schützenberger. The proof of the general formula is easier than those described here, in that it requires only a knowledge of $\mathbb{P}^1$-bundles in place of the calculations of Gysin formulas. For details, see

W. Fulton, *Flags, Schubert polynomials, degeneracy loci, and determinantal formulas*, Duke Math. J. **65** (1992), 381–420.

9. (p. 51) For this, see

P. Pragacz, *Cycles of isotropic subspaces and formulas for symmetric degeneracy loci*, in *Topics in Algebra*, Banach Center Publications, vol. 26, part 2, 1990, pp. 189–199.

10. (p. 52) As in Note 8, these formulas have become part of a more general story of degeneracy loci. For each of the classical groups, there is such a locus for each element in the corresponding Weyl group. For this, see

> W. Fulton, *Determinantal formulas for orthogonal and symplectic degeneracy loci*, to appear in J. Diff. Geom.
>
> W. Fulton, *Schubert varieties in flag bundles for the classical groups*, to appear in *Proceedings of Conference in Honor of Hirzebruch's 65th Birthday*, Bar Ilan, 1993, Amer. Math. Soc.
>
> P. Pragacz and J. Ratajski, *Formulas for Lagrangian and orthogonal degeneracy loci; the $\widetilde{Q}$-polynomials approach*, preprint.

The second reference includes the deduction of the general case from the case when L is a square; as Totaro points out, this deduction is not as simple as had been thought, since there is no "squaring principle" for line bundles that includes 2-torsion.

11. (p. 54) In case one is intersecting with divisors in one linear system, it is possible to find a further refinement of these intersection products, at a possible cost of extending the ground field. For the strongest results in this direction, see

> L. van Gastel, *Excess intersections and a correspondence principle*, Invent. Math. **103** (1991), 197–211.

Vogel and his coauthors have continued to study the refinements of Bézout's theorem. For example, see

> H. Flenner and W. Vogel, *Improper intersections and a converse to Bezout's theorem*, J. of Algebra **159** (1993), 460–476.

In case the ambient variety is projective space, the paper of van Gastel includes an explanation of how to translate between the constructions of Vogel and the intersection theory described in this book.

12. (p. 55) In fact, all of the conics can be real! We discovered this in 1986, but did not publish a proof. Recently a detailed proof has been given:

> F. Ronga, A. Tognoli, and T. Vust, *The number of conics tangent to five given conics: the real case*, preprint.

F. Sottile, in his 1994 University of Chicago PhD thesis, proved analogous results for intersections of Schubert cycles in any Grassmannian of lines in any projective space. The methods in all cases are by explicit deformations. It is intriguing to speculate about how general this phenomenon is, when the problem is one of counting how many figures of some kind have a given position with respect to some given general figures.

13. (p. 56) The general case of this has now been proved:

S. Kleiman, J. Lipman, and B. Ulrich, *The source double-point cycle of a finite map of codimension one*, in *Complex Projective Geometry*, London Math. Soc. Lecture Note Series **179** (1992), 199–212.

14. (p. 57) For more about multiple point formulas, see

S. Kleiman, *Multiple point formulas I: iteration*, Acta Math. **147** (1981), 13–49.

S. Kleiman, *Multiple point formulas II: the Hilbert scheme*, in *Enumerative Geometry (Sitges, 1987)*, Springer Lecture Notes in Math. **1436** (1990), 101–138.

15. (p. 61) For a generalization, see

W. Fulton, *Positive polynomials for filtered ample vector bundles*, Amer. J. Math. **117** (1995), 627–633.

16. (p. 68) For a deduction of the singular case from the nonsingular case, see

B. Angeniol and F. El Zein, *Théorème de Riemann-Roch par désingularisation*, Bull. Sci. Math. France **116** (1988), 385–400.

17. (pp. 68, 74) P. Roberts has used these ideas, especially the graph construction, to prove part of a conjecture of Serre about the vanishing of the intersection number in local algebra:

P. Roberts, *Local Chern characters and intersection multiplicities*, in *Algebraic Geometry, Bowdoin 1985*, Proc. Sympos. Pure Math. **46** part 2, Amer. Math. Soc., 1987, pp. 389–400.

An independent proof was also given by Gillet and Soulé using K-theory and Adams operations:

H. Gillet and C. Soulé, *Intersection theory using Adams operations*, Invent. Math. **90** (1987), 243–277.

More on the graph construction can be found in:

H. Gillet and C. Soulé, *An arithmetic Riemann-Roch theorem*, Invent. Math. **110** (1992), 493–543.

18. (p. 70) There has been considerable progress on the relations between cycles and intermediate Jacobians. For example:

C. Voisin, *Une approche infinitesimal du théorème de H. Clemens sur les cycles d'une quintique génerale de $\mathbb{P}^4$*, J. Algebraic Geometry **1** (1992), 157–174.

M. Nori, *Algebraic cycles and Hodge-theoretic connectivity*, Invent. Math. **111** (1993), 349–373.

N. Suwa, *Sur l'image de l'application d'Abel-Jacobi de Bloch*, Bull. Sci. Math. France **116** (1988), 69–101.

H. Esnault and M. Levine, *Surjectivity of cycle maps*, in *Journées de Géométrie Algébrique d'Orsay*, Asterisque **218** (1993), 203–216.

19. (p. 72) Totaro has shown by examples why this cannot exist in general:

B. Totaro, *Chow groups, Chow cohomology, and linear varieties*, to appear in J. Alg. Geom.

20. (p. 73) Kleiman and Thorop have given some variations on this theme, in section 3 of

S. Kleiman, *Intersection theory and enumerative geometry; a decade in review*, Proc. Symp. Pure Math. Amer Math Soc. **46** (2), 1987, pp. 321–370.

Practical methods for calculating these groups have also been given:

S. Kimura, *Fractional intersection and bivariant theory*, Communications in Algebra **20** (1992), 285–302.

Kimura's paper also explains how rational intersection numbers for curves on normal surfaces can be interpreted by means of these operational Chow cohomology groups. For another approach to Chow cohomology, see

A. Suslin and V. Voevodsky, *Relative cycles and Chow sheaves*, preprint.

21. (p. 74) There has been some progress on this question:

G. Barthel, J.-P. Brasselet, K.-H. Fieseler, O. Gabber, and L. Kaup, *Relèvement de cycles algébriques et homomorphismes associés en homologie d'intersection*, Ann. of Math. **141** (1995), 147–179.

22. (p. 74) This development has taken place and is continuing. A general theory has been developed:

H. Gillet and C. Soulé, *Arithmetic intersection theory*, Inst. Hautes Études Sci. Publ. Math. **72** (1991), 94–174.

For a survey, with references, see

C. Soulé, D. Abramovich, J.-F. Burnol, and J. Kramer, *Lectures on Arakelov Geometry*, Cambridge Studies in Advanced Mathematics, vol. 33, Cambridge University Press, 1992.

These notes have only described some recent work in intersection theory as it relates to topics discussed in the book. There have also been several important developments that go well beyond what was envisioned in 1984. We mention only a few of these, with a small sampling of references:

Bloch's higher Chow groups, and relations to higher K-theory and Beilinson's regulator:

S. Bloch, *Algebraic cycles and higher K-theory*, Adv. in Math. 61 (1986), 267–304.

C. Deninger, *The Beilinson conjectures*, in *L-functions and Arithmetic (Durham, 1989)*, London Math. Soc. Lecture Note Series **153** (1991), 173–209.

H. Esnault and E. Viehweg, *Deligne-Beilinson cohomology*, in *Beilinson's Conjectures on Special Values of L-functions*, Perspectives in Mathematics, vol. 4, Academic Press, 1988, pp. 43–92.

Motives and Chow groups:

J.-P. Murre, *On the motive of an algebraic surface*, J. Reine Angew. Math. **409** (1990), 190–204.

U. Jannsen, *Motivic sheaves and filtrations on Chow groups*, in *Motives (Seattle WA 1991)*, Proc. Sympos. Pure Math. **55**, part 1, 1994, pp. 245–302.

V. Voevodsky, *Triangulated categories of motives over a field*, preprint.

A theory, based on homotopy groups of Chow varieties, called Lawson homology, that interpolates between ordinary homology and Chow groups of algebraic varieties:

H. B. Lawson, *Algebraic cycles and homotopy theory*, Ann. of Math. **129** (1989), 253–291.

E. M. Friedlander and H. B. Lawson, *A theory of algebraic cocycles*, Ann. of Math. 136 (1992), 361–428.

Intersection theory on moduli spaces, especially as influenced by physics:

E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, in *Surveys in Differential Geometry (Cambridge, MA, 1990)*, Lehigh Univ., 1991, pp. 243–310.

M. Kontsevich, *Intersection theory on the moduli space of curves*, Funct. Anal. and Appl. **25** (1991), 123–129.

E. Looijenga, *Intersection theory on Deligne-Mumford compactifications (after Witten and Kontsevich)*, Séminaire Bourbaki, Exp. 768, 1992–93, Asterisque **216** (1993), 187–212.

Quantum cohomology, with applications to enumerative geometry:

M. Kontsevich and Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys. **164** (1994), 525–562.

P. Di Francesco and C. Itzykson, *Quantum intersection rings*, preprint.

W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, preprint.

Kleiman and Thorup (see Note 20) defined the notion of an Alexander scheme. The Chow groups of such a variety, at least after tensoring with $\mathbb{Q}$, have a natural ring structure. These are now quite well understood:

A. Vistoli, *Alexander duality in intersection theory*, Compositio Math. **70** (1989), 199–225.

S. Kimura, *On the characterization of Alexander schemes*, Compositio Math. **92** (1994), 273–284.

Most of the intersection theory described here has been extended to Deligne-Mumford stacks: See Vistoli (Note 5) and

H. Gillet, *Intersection theory on algebraic stacks and Q-varieties*, J. Pure and Appl. Algebra **39** (1984), 193–240.

Finally—with apologies to the many whose papers should be included in such a list—a few other papers that may be of interest to readers of these notes:

S. Bloch, M. P. Murthy, and L. Szpiro, *Zero-cycles and the number of generators of an ideal*, Mém. Soc. Math. France **38** (1989), 51–74.

J.-P. Demailly, *Monge-Ampère operators, Lelong numbers and intersection theory*, Complex Analysis and Geometry, Plenum, 1993, pp. 115–193.

S. Keel, *Intersection theory of linear embeddings*, Trans. Amer. Math. Soc. **335** (1993), 195–212.

X. Wu, *Residual intersections and some applications*, Duke Math. J. **75** (1994), 733–758.

P. Aluffi, *Singular schemes of hypersurfaces*, preprint.

There have been many interesting and important papers on enumerative goemetry besides those mentioned in these notes. Some of these can be found in Kleiman's survey in Note 20, but it would take another volume to describe the work in this area during the succeeding decade.

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This book presents expository lectures from the CBMS regional conference held at George Mason University during the summer of 1983. This volume has been reprinted by the AMS with updates and corrections. In the work, Fulton gives references to many further developments in the field.

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