Gilles Pisier

FACTORIZATION OF LINEAR OPERATORS AND GEOMETRY OF BANACH SPACES

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Introduction

In 1956 Grothendieck published a fascinating paper entitled Résumé de la théorie métrique des produits tensoriels topologiques. This paper, which is now referred to as “the Résumé”, has had a considerable influence on the development of Banach space theory since 1968. It contained a general theory of tensor norms on tensor products of Banach spaces, described several operations to generate new tensor norms from some known ones, and studied the duality theory of these norms. But the highlight of the Résumé is a result that Grothendieck called “the fundamental theorem of the metric theory of tensor products” and which is now called Grothendieck’s theorem (or sometimes Grothendieck’s inequality). Among its many consequences, it implies that every bounded operator from $L_\infty$ into $L_1$ factors through $L_2$. This theorem remained practically unnoticed until 1968, when Lindenstrauss and Pełczyński revived it and gave a detailed proof (cf. [L-P]). Although there are now numerous simple proofs of this theorem (cf. e.g. Chapter 5), it remains a nontrivial result.

The aim of the present lecture notes is to describe the contributions made since 1968 in the directions opened by the Résumé. Although our title is very general, we will limitate ourselves to the work which is directly related to the questions raised in Grothendieck’s paper. The Résumé ends with a list of six problems with comments on each of them. Thanks to the considerable progress achieved in Banach space theory in the last 15 years, these problems are now all solved (except perhaps for the exact value of the Grothendieck constant), and these lecture notes will include the various results which led to their solution. These six problems are actually all linked together and related to several central questions. To summarize simply the contents of these notes, we might say that they revolve around the following questions: When does an operator $u : X \to Y$ (between two Banach spaces) factor through a Hilbert space? For which spaces $X, Y$ does this happen for all operators $u$? We will examine the particular case of operators defined on a Banach lattice, a $C^*$-algebra, or the disc algebra and $H^\infty$.

The topics that we cover have many connections or applications outside Banach space theory, and we hope that they will have even more in the future. With this in mind, we have tried to make this material accessible to nonspecialists, so that our redaction is usually quite detailed and self-contained. For the same
reason, we have deliberately kept to a minimum the use of the duality theory via
the trace, since we feel that this might turn off the readers who are not familiar
with it. Nevertheless, we urge the readers who want to go deeper in the theory to
get acquainted with the principles of this duality (cf. [P1 or Pe4]). We should
mention that our restricted selection has left out several important topics. We
refer to [P1] for the general theory of operator ideals which was developed by
Pietsch and his school since the late sixties. The characterization of $L_p$-spaces (or
subspaces of $L_p$ or subspaces of quotients of $L_p$) by operator theoretic properties
is a major omission. For this, we refer the reader to the beautiful paper of
Kwapień [Kw3] and to its references. Also, the factorization theorems of Maurey
(and the important work of Rosenthal [R2]) are not included here; we refer the
reader to [M1]. We do discuss, however, the general theory of type and cotype,
but briefly and without proofs. We will be mainly concerned here with type 2 or
cotype 2. In general, we have concentrated on the problem of factoring an
operator through $L_2$, and we have left out the natural extensions for the
factorization through $L_p$. In our exposition, we will come across most of the line
of investigation which forms the so-called local theory of Banach spaces—i.e., the
study of Banach spaces by finite-dimensional methods. We have tried to indicate
in the references, as often as possible, the ramifications of this currently very
active area.

Let us now review the contents of these notes. In Chapter 0, we introduce the
projective and injective tensor products and the approximation property (in short
A.P.). Among the six problems at the end of the Résumé, the first and most
famous one was the approximation problem: Does every Banach space possess
the A.P.? Enflo [E] gave a counterexample in 1972, which opened a new era in
functional analysis.

In Chapter 0, we have insisted on the necessary distinction between nuclear
operators and elements of the projective tensor product, which is essential in
Chapter 10.

In Chapter 1, we present in detail the basic theory of $p$-summing operators and
its first applications to Banach space theory: For every $n$-dimensional subspace $E$
of a space $X$, there is a projection $P: X \to E$ such that $\|P\| \leq \sqrt{n}$ and an
isomorphism $T: l_2^n \to E$ such that $\|T\|\|T^{-1}\| \leq \sqrt{n}$.

In §6, we briefly introduce $p$-integral operators and some rudiments of duality
theory, but this is not used in the sequel. We note in passing that the Radon-
Nikodým property (which is crucial to compare integral and nuclear operators) is
not discussed at all here; we refer the reader to [D-U] for this topic. In Chapter 2,
we give the Lindenstrauss-Pelczyński criterion for an operator to factor through a
Hilbert space. This can be viewed as an application of the Hahn-Banach theorem
provided a certain duality theorem is explicitied; we do this in §2.b. In Chapter 3,
we introduce the notions of type and cotype and prove Kwapień’s theorem that
every space of type 2 and of cotype 2 is isomorphic to a Hilbert space. The theory
of type and cotype provides a useful scale to measure how close a given space is
from a Hilbert space. We briefly review the main points of this theory in §3.3 (we use only the extreme cases of type 2 or cotype 2 in the sequel). In Chapter 4, we prove a factorization theorem which links Kwapieński’s theorem and Grothendieck’s theorem: If $X^*$ and $Y$ are of cotype 2, then every approximable operator from $X$ into $Y$ factors through a Hilbert space. This result plays a crucial rôle in the construction of Chapter 10. As an application, in §4.b, we show that Sidon sets in the dual of a compact Abelian group $G$ are characterized by the fact that they span a cotype 2 space in $C(G)$. This generalizes an earlier result of Varopoulos [VI]. In Chapter 5, we concentrate on Grothendieck’s theorem, which we abbreviate G.T. Chapter 5 contains at least four proofs of that theorem. In §5.a, we briefly introduce $L_p$-spaces (there is more information in §§8.b and 8.c). We are mainly concerned here with the cases $p = 1$ and $p = \infty$. This allows us to state and prove G.T. in the framework of [L-P]: Every operator from an $L_1$ space into an $L_2$ space is 1-summing. This is proved in §5.c. In §5.b we give the (somewhat dual) formulation about operators defined on a $C(K)$-space or on an $L_\infty$-space. We tried to give explicitly all the various forms in which the theorem can be used, and we distinguished carefully between the easy part (which we call the “little G.T.”) and the more delicate part of this theorem. We first give a proof derived from the more “abstract” result of Chapter 4, but §5.d contains another proof, more direct and of independent interest.

In §5.3, we include Krivine’s proof of G.T., which gives the best known upper bound for the constant $K_G$. In problem 3 in the Résumé, Grothendieck asked for the exact value of various constants (see 5.3 for details); this is the only problem which is not completely solved (but of course, it is probably the least important one!). In 5.f, we give a very quick proof of G.T., based on a property of the space $H^1$, due to Pelczyński and Wojtaszczyk.

In Chapter 6, we study the Banach spaces satisfying G.T., which we call G.T. spaces. We include several characterizations of these spaces, but we insist more on the a priori smaller class of G.T. spaces of cotype 2. The latter enjoys nicer stability properties and includes all the known examples of G.T. spaces. In 6.c, we show that if $R$ is a hilbertian (or, more generally, a reflexive) subspace of $L_1$, then $L_1/R$ is a G.T. space of cotype 2.

We come here to problem 5 in the Résumé. A stronger formulation of this problem was given by Lindenstrauss and Pełczyński, who asked whether the $\mathcal{L}_1$-spaces are the only spaces satisfying G.T. The above result of 6.c (due to Kisliak and the author) gives a negative answer to this question (and a fortiori to problem 5) since the quotients $L_1/R$ are never $\mathcal{L}_1$-spaces when $R$ is a reflexive infinite-dimensional subspace of $L_1$.

It is rather easy to give concrete examples of hilbertian subspaces of $L_1$: for example, the span of the Rademacher functions. However, it is more delicate to produce “very large” such spaces. For this purpose, we present in Chapter 7 a method based on volume estimates which yields an orthogonal decomposition of $l_1^{2^n}$ into two parts which are uniformly (with respect to $n$) isomorphic to $l_2^n$. This
result originates in the work of Kašin, but the method was developed in [Sz2] and
[S-T].

We also give in §8.g an infinite-dimensional version of this decomposition, obtained recently by Krivine.

In Chapter 8, we turn to Banach lattices and start by a reformulation of G.T. in this context. In 8.b, we introduce ultraproducts with several simple illustrative applications. Problem 2 in the Résumé asked whether a specific property (involving tensor norms) was always satisfied. This was answered negatively by Gordon-Lewis [G-L1]. Their paper showed that this property (now called the G.L. property) provides a useful criterion to decide whether or not a given space is isomorphic to a Banach lattice (or more generally to a space with l.u.st.). This is the subject of 8.c; in 8.d, we show that, for \( p \neq 2 \), the Schatten classes \( C_p \) do not have the G.L. property (cf. [G-L1, Se]). Many more spaces without l.u.st. are now known. Moreover, one can construct, for any \( n \), an \( n \)-dimensional space with l.u.st. constant greater than \( \delta \sqrt{n} \), for some \( \delta > 0 \) independent of \( n \). This “worst possible” case can be exhibited in 8.e very quickly (following [F-K-P]), using Chapter 7. In §8.g, we show (following [L-P]) that an atomic Banach lattice which satisfies G.T. must be isomorphic to \( l_1(\Gamma) \) for some set \( \Gamma \).

In Chapter 9, we present the \( C^* \)-algebraic version of G.T., as conjectured by Grothendieck. Here we mainly follow [Pi7] and Haagerup’s work [H1]. This was problem 4 in the Résumé. In §7.b, we discuss (without proofs) several applications of these results to the theory of derivations and representations of \( C^* \)-algebras (cf. [Bu, C1, C2, H2, H3]).

Finally, in Chapter 10, we construct (following [Pi10]) several Banach spaces \( X \) such that \( X \otimes X = X \otimes X \). This gives a negative solution to the sixth and last problem in the Résumé. Grothendieck conjectured there that this could happen only in the finite-dimensional case. The reader who has reached this point will be rewarded to find that all the results used in the construction have been included (with complete proofs) in the preceding chapters (mainly in Chapters 4, 7 and 6.c).

Each chapter is followed by a notes and references section where the reader will find the credits for the corresponding results, as well as some additional comments. In general, we give references in the text itself for the statements which we quote without proof.

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