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Introduction

Harmonic analysis on symmetric spaces is for me a very inspiring combination of analysis, geometry and algebra. In this book I shall try to present this subject with special emphasis on those pseudo-Riemannian symmetric spaces which have a semisimple group of isometries. We shall call these the semisimple symmetric spaces.

Harmonic analysis on Riemannian semisimple symmetric spaces is very well established, primarily through the work of H. Weyl, E. Cartan, Harish-Chandra and S. Helgason.

Among the non-Riemannian semisimple symmetric spaces are, for example, the noncompact semisimple groups and the hyperbolic spaces. For these special examples of non-Riemannian symmetric spaces there is also a well-established harmonic analysis. However, for the general semisimple symmetric spaces, harmonic analysis is far less developed, and many basic questions have not yet found a final answer.

My own contribution to this subject is primarily the idea of how to construct the discrete series for such a space. I hope I am excused for putting some emphasis on this aspect. In [**c**], where I first presented the construction, I tried to show that the construction is very elementary and direct. In this book I have chosen to let the general ideas behind the construction play a fundamental role—that is, the duality principle and the orbit picture related to it and also the definition of representations by means of distributions on the orbits. At the same time I have tried to give a rather systematic treatment of the basic problems in harmonic analysis on symmetric spaces and to discuss some of the more important recent developments in the theory.

There are a few new results in the text. In Example B of Chapter III there is a new and simple proof of the Paley-Wiener theorem for Riemannian symmetric spaces of the noncompact type. In §3 of Chapter IV it is proved that any "H-finite" joint eigenfunction on a Riemannian symmetric space is the Poisson transform of a distribution on the boundary. This result implies that we, to a large extent, can avoid mentioning hyperfunctions in our construction of representations.

In Chapters VI and VII several results are generalized, and there are indications of simplifications of proofs compared to the existing literature.

To follow the presentation, at least for the later chapters, I think that the reader should have some familiarity with the basic structure theory of semisimple. Lie groups and Lie algebras. After each chapter I have included some very brief notes indicating related results, historical aspects or references to proofs not mentioned in the text.

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