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Extreme Graph Theory with Emphasis on Probabilistic Methods

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Preface

This monograph is based on the marathon lecture series given at the NSF-CBMS Regional Conference on Extremal Graph Theory held in June 1984 at Emory University, Atlanta. The author is grateful to Dwight Duffus, Ron Gould, and Peter Winkler for their superb organization of the meeting; the additional lectures by Dick Duke, Ralph Faudree, Ron Graham, and Tom Trotter greatly enriched the conference.

Since the publication of the author's book, *Extremal Graph Theory* (Academic Press, London, New York, and San Francisco, 1978, to be referred to as EGT), a number of important results have been proved, and one of the aims of the lectures was to update EGT by presenting some of these developments.

Over the past few years a noticeable shift has been taking place in extremal graph theory towards probabilistic methods. The most obvious sign is that random graphs are used more and more, but that is not all. Even more significantly, a probabilistic frame of mind was needed to find many of the proofs, which on the surface have nothing to do with probabilistic ideas. In several beautiful and difficult proofs the underlying philosophy is that we do not have to care about single vertices, say, for it suffices to make use of the fact that there are many subsets of vertices of a given cardinality with the right properties. To give a simple example, one often makes use of the fact that if $X_1, X_2, \dots, X_N$ are nonnegative integers bounded by A, $\sum_{i=1}^{N} X_i = Na$ and $0 < b < a$, then at least $(a-b)N/(A-b)$ of the X_i's are greater than b. Equivalently, if X_i is a random variable, $0 < X < A$ and $E(X) = a$, then

$$P(X > b) \geqslant (a-b)/(A-b) \quad \text{for all } 0 < b < a. \tag{1}$$

Inequality (1) has the following reformulation in graph-theoretic terms. If B is a bipartite graph with bipartition (X, Y), $X = \{x_1, x_2, \dots, x_m\}$, $Y = \{y_1, y_2, \dots, y_n\}$, $d(y_j) \leqslant \Delta$ for all j, $1 \leqslant j \leqslant n$, then for $d' < d = \sum_{i=1}^{m} d(x_i)/n$ there are at least $(d-d')n/(\Delta - d')$ vertices y_j of degree at least d'.

Needless to emphasize, in the great majority of the cases the merit is in finding the need for probabilistic inequalities and applying them cleverly, and not in proving the inequalities. The main aim of the lecture was to show how fruitful a probabilistic frame of mind is in tackling main line extremal problems.

The notation in these notes is taken from EGT. In particular, $|G|$ is the order of a graph G, i.e. the number of vertices, and $e(G)$ is the size of G, i.e. the number of edges. The cardinality of a set U is denoted by $|U|$, and the collection of r-subsets of U is $U^{(r)}$. Though these notes are practically self-contained, familiarity with at least some parts of EGT will certainly help the reader. A conscious effort has been made to prevent the lectures from turning into a long catalogue of results; without this effort, the monograph could have ended up with several hundred results. However, there seems little doubt that it is much more useful to present just a few of the deeper results and thereby leave time to dwell on the proofs.

The first two sections are closely related. They deal with subdivisions of graphs and subcontractions. Both areas owe a considerable amount to Mader, who proved that for every $p \in \mathbf{N}$ there are constants $s(p)$, $c(p)$ such that every graph of order n and size greater than $s(p)n$ contains a topological complete graph of order p, and every graph of order n and size greater than $c(p)n$ has a subcontraction to K^p. (Needless to say, $s(p)$ and $c(p)$ are taken to be the smallest values that will do in the statements above.) Consequently for every fixed graph H there is a constant $s(H)$ such that every graph of order n and size greater than $s(H)n$ contains a subdivision of H and there is an analogous constant $c(H)$. Bollobás started the study of subdivisions of graphs with some constraints on the subdivisions one allows. For example, we may wish to restrict the number of times we subdivide an edge, at least modulo some integer k. The main aim of §1 is to present recent result of Thomassen in this area, with a considerably better bound than the original one given by Thomassen.

The second section, on subcontractions, is devoted to a new result of Thomason and Kostochka, improving the upper bound on $c(p)$ proved by Mader. Together with a rather easy result of Bollobás, Catlin, and Erdős, this result implies that the order of $c(p)$ is $p(\log p)^{1/2}$, a fact not many of us would have expected.

The third and fourth sections concern different aspects of essentially the same problem. At least how many vertices must we have if the minimal degree is δ and the girth is at least g? At most how many vertices can we have if the maximal degree is at most Δ and the diameter is at most D? An "ideal" graph would answer both questions, but the trouble is that there are very few such ideal graphs. One is left with approximating the appropriate functions either by constructing suitable functions or by showing, usually by probabilistic methods, that suitable graphs do exist. As far as the bounds are concerned, the nonconstructive methods due to Erdős, Sachs, Bollobás, de la Vega, and others give better results, but the constructions have obvious advantages. In these sections the emphasis is on new constructive methods due to Bermond, Delorme, Farhi, Leland, Solomon, Jerrum, Skyum, and Margulis.

In §5 we concentrate on a substantial recent result of Gyárfás, Komlós, and Szemerédi concerning the distribution of cycle lengths in graphs with fairly many edges. Though the theorem is interesting, it is the proof, rich in ideas and techniques, that really justified spending two lectures on the result.

The sixth section contains a telegraphic review of the theory of random graphs. The highlights are the classical theorem of Erdős and Rényi on the evolution of random graphs and its recent extensions due to Bollobás.

In §7 we present a surprising and beautiful result of Beck on size Ramsey numbers. As a simple application of random graphs, Beck proved that there are graphs $G_1, G_2, \ldots$ such that G_i has at most ci edges, where c is a constant, and in any coloring of the edges of G_i with two colors we can find a monochromatic path of length s.

Saturated graphs were introduced over twenty years ago by Erdős, Hajnal, and Moon. Their result was extended considerably by Bollobás, who also introduced weakly saturated graphs. The main conjecture concerning weakly saturated graphs was proved recently by Alon, Frankl, and Kalai; the simple and elegant proof, based on exterior products (!), is presented in §8.

The last section, §9, concerns restricted colorings of graphs. We know from Vizing's theorem that a graph of maximal degree Δ is $(\Delta + 1)$-colorable. What happens if we prescribe a list for each edge from which the color of the edge has to be chosen? What is the maximal length of the lists that always let us color our graph? It has been conjectured that lists of length $\Delta + 1$ will do. This conjecture, if true, would clearly be best possible.

At the moment, the conjecture is far from being proved and many graph theorists suspect it to be false. The main aim of the section is to present a recent result of Bollobás and Harris, implying that for some constant $c < 2$ lists of size at most $c\Delta$ will do for every graph of maximal degree $\Delta \geqslant 3$. As a trivial consequence of this result, Bollobás and Harris made the first substantial progress towards a proof of a long-standing conjecture of Behzad concerning the total chromatic number.

It is a pleasure to thank Fan Chung, Dwight Duffus, Ron Graham, Hal Kierstead, and Andrew Thomason for their ideas and suggestions, many of which have been incorporated into the text. Finally, I would like to express my gratitude to all participants of the conference for their enthusiasm for the subject and the warm reception of the lectures.

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