Conference Board of the Mathematical Sciences

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Regional Conference Series in Mathematics

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Towards the end of this meeting I became quite ill. I wish to express my appreciation to all the participants of the conference, especially to Vic Camillo, Frank Hannick, Jeanne Kerr and Sylvia Wiegand, for their kindness and concern. I would also like to thank Susan Montgomery for giving my last four lectures when I was unable to do so.

Finally, love to my family, Marj, Barbara and Jon, for all things nonmathematical.

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Introduction

During the week of June 17, 1985 a conference was held at Mankato State University in Mankato, Minnesota. This monograph contains slightly expanded versions of the ten talks I wrote for the occasion. Their theme is the interplay between group theory and ring theory. Specifically, they concern group rings, crossed products and the Galois theory of rings.

Group rings. The group ring $R[G]$ is an easily defined, attractive object to study. It is a tool of group theory, a tool of ring theory and an interesting subject in its own right. If G is a finite group and R is a field, then the subject concerns the ordinary and modular representations and characters of G. If R is the ring of integers, then we are dealing with the integral representations of G and its close relationship with maximal orders and algebraic number theory.

If G is polycyclic-by-finite and R is a Noetherian ring, then so is $R[G]$. Thus we obtain a large class of interesting examples of such rings, and indeed this subject is now closely allied with the general study of noncommutative Noetherian rings. Furthermore, there are many analogies here with enveloping algebras of finite-dimensional Lie algebras.

For G arbitrary, it is best to restrict our attention to R being a field K or the ring of integers. However, even with this assumption, we can no longer use finite techniques and the problems become correspondingly more difficult. The subject of group algebras of infinite groups was begun in the 50's and 60's with work of G. Higman, I. Kaplansky and S. A. Amitsur. It was most active in the 70's. The first four lectures are devoted to this material.

Crossed products. In some sense the crossed product $R * G$ is a generalized group ring, but it is actually much more. It first occurred in the study of division rings and, in particular, of the Brauer group of fields. A classical question with a modern answer due to Amitsur concerned whether all finite-dimensional division rings are crossed products. In another vein, if one is interested in the Galois theory of rings, then the skew group ring RG is an important tool. It of course contains all the ingredients of the theory, namely the ring R, the group G and the fixed ring R^G. Results on crossed products have had impressive Galois theoretic applications.

Furthermore, even if one is only interested in ordinary group algebras $K[G]$, then crossed products still occur. For example, if N is a normal subgroup of G, then $K[G] = K[N] * (G/N)$ is a suitable crossed product of the quotient group G/N over the subring $K[N]$. In addition, it frequently happens that if P is a prime ideal of $K[G]$, then $K[G]/P$ has a natural crossed product structure. For these and many other reasons, crossed products are interesting and important objects of study. Three lectures are devoted to this subject, emphasizing fairly recent work.

Galois theory. The Galois theory of noncommutative rings is a new subject with roots in invariant theory and the classical Galois theory of fields and division rings. If G acts on a ring R, then we are concerned with the relationship between R and the fixed ring R^G, and the relationship between the subgroups of G and the intermediate rings.

One can argue that the subject began in the early 70's with the important Bergman-Isaacs theorem on the existence of fixed points. This was later extended by V. K. Kharchenko who also developed a general Galois theory for semiprime rings. Important contributions were made by J. W. Fisher, S. Montgomery, J. Osterburg and many others. Actions on specific rings have also been studied. Of particular interest are actions on free rings, generic matrix rings, polynomial rings, enveloping algebras and group algebras. The entire subject is quite active at present and is being pursued in many different directions. The final three lectures consider selected aspects of this material.

The ten talks included here are essentially independent although the notation builds up. Each contains a selection of results on a particular subject, a limited number of proofs or sketches and at least a few open questions. Furthermore, each has its own brief list of references. Two general references for the series as a whole are S. Montgomery's monograph, *Fixed Rings of Finite Automorphism Groups of Associative Rings*, and my book, *The Algebraic Structure of Group Rings*. Finally, since the topics chosen are all part of a common theme, they do begin to merge and the lectures are eventually all interrelated.

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