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Preface

This monograph is an expanded version of a CBMS series of lectures delivered in Miami in January, 1984. As in the lectures, our goal is to provide an introduction to minimax methods in critical point theory and their application to problems in differential equations. The presentation of the abstract minimax theory is essentially self-contained. Most of the applications are to semilinear elliptic partial differential equations and a basic knowledge of linear elliptic theory is required for this material. An overview is given of the subject matter in Chapter 1 and a detailed study is carried out in the chapters that follow.

Many friends have contributed to my study and organization of this material. I thank in particular Antonio Ambrosetti, Abbas Bahri, Vieri Benci, Henri Berestycki, Haïm Brezis, Michael Crandall, Edward Fadell, Suffian Husseini, Jürgen Moser, and Louis Nirenberg for their inspiration, encouragement, and advice. The CBMS conference was hosted by the Mathematics Department of the University of Miami. Further thanks are due to the members of the department, especially to Shair Ahmad and Alan Lazer for their efficient handling of the meeting and their kind hospitality.

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