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George E. Andrews

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To
Joy, Amy, Katy, and Derek

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Preface

In recent years, q-series have popped up in work in physics, Lie algebras, transcendental number theory, and statistics, in addition to new developments in areas more familiar with q-series: classical analysis, combinatorics, and additive number theory. The immense amount of activity taking place makes it impossible to give a comprehensive account in ten chapters. Obviously the areas I choose to emphasize will be a matter of my own taste and reflect to some extent those projects that have most engaged my own interests and efforts. I have tried to include some accounting of major breakthroughs (e.g. Chapters 4 and 5) even though my own contributions may have been marginal.

There have been numerous serendipitous events that have marked the development of q-series in the last one hundred years. I try to discuss a few of these in Chapter 1. Chapters 2 and 3 might be called Rogers and Bailey revisited. In these two chapters I combine history with modern work. I redo Rogers's first proof of the Rogers-Ramanujan identities with the help of a modern symbolic manipulation computer language like SCRATCHPAD. Bailey's greatest discovery, Bailey's Lemma, is examined and used to set up the mathematical tools required for a complete treatment of recent extensions of the Hard Hexagon Model.

In Chapter 4 we examine recent work on constant term problems. The great recent achievement here is the Zeilberger-Bressound proof of the q-Dyson conjecture. While it is impossible to give their proof, we can at least provide some relevant background. Chapter 5, Integrals, has obviously many common elements with Chapter 4. The most important aspect of this work is the re-emergence of Selberg's integral. We provide a full discussion following R. Askey's approach.

Chapters 6 and 7 are devoted to recent developments in additive number theory. Work of Bressoud, Garsia and Milne is discussed in Chapter 6 culminating in the Garsia-Milne Involution Principle. Chapter 7 presents generalized Frobenius partitions and the recent contributions of Garvan and Kolitsch. Chapter 8 is an expository introduction to Baxter's solution of the Hard Hexagon Model. I try to indicate how the machinery from Chapter 3 applies to this work and its extensions.

Chapter 9 looks at results closely related to Ramanujan's "Lost" Notebook. While few results due directly to Ramanujan are presented, the discussion is

primarily motivated, nonetheless, by an attempt to unify work from the "Lost" Notebook (see Chapter 1 and Andrews [27]).

We conclude with a discussion of how symbolic manipulation languages such as SCRATCHPAD can be utilized to further research on q-series.

As is clear I was helped directly and indirectly by the work of many in preparing this book. I wish to single out especially Richard Askey and Rodney Baxter. Chapter 6 especially relies heavily on Askey's work including his own not widely accessible proof of the Selberg integral. Much of the introductory material in Chapter 8 is taken from an expository lecture given by Baxter at Penn State in 1980. Also thanks are due to IBM (in particular, Dick Jenks and David Yun) who provided SCRATCHPAD to Penn State in a field test; four years ago computer algebra played only a minor part in my work. Now I do not know how I could get along without it.

I owe a great debt to many people who were involved with the NSF-CBMS Regional Conference at Arizona State University, in May 1985, where these lectures were presented. Mourad Ismail and Ed Ihrig organized the conference well and made it run quite smoothly. Kevin Kadell provided reliable transportation whenever I needed it. Richard Askey, whose ideas have had such importance in recent q-series research, helped in numerous ways to make the conference work. Bonnie Randolph capably and expeditiously typed the manuscript. To all these people, many thanks.

For special thanks I must single out my wife, Joy. Besides putting up with me during the several months preceding the conference, when I thought of little else, she also, at the last minute, averted my absence from the conference. On the Sunday afternoon before my initial lecture at nine o'clock Monday morning, she and I arrived at the University Park Airport to find that my flight to Pittsburgh was cancelled. There following her rapid unscheduled drive to Pittsburgh combined with creative double-parking and baggage management par excellence. Because of the United Airlines' strike, I would surely not have gotten to Tempe for several days if I had not caught that flight. As it was I was five minutes late, but the flight was delayed twenty minutes.

University Park, Pennsylvania
July 1985

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