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Contents

Introduction

The present work is intended to develop three main results:

(1) An extension of the standard basis theorem, going back to Doubilet, Rota, and Stein, and eventually to Capelli and Young, to algebras containing positively signed and negatively signed variables, or superalgebras as we call them. Such an extension has required a rethinking of some of the basic concepts of linear algebra, such as "matrix" and "coordinate system," along lines that we believe to be new, and which we hope will lead to an extension to "signed" modules of the entire apparatus of linear algebra. The standard basis theorem, which we prove, is characteristic-free and includes, besides the classical case, straightening algorithms, which apply to permanents as well as to determinants, as well as a mixed generalization of the notion of both determinant and permanent, called the biproduct, which differs from the Berezin determinant.

(2) A rigorous presentation of the symbolic method of invariant theory for symmetric tensors, in characteristic zero. The results here offer no great novelty over the nineteenth century, except rigor.

(3) A new symbolic method (foreshadowed by Weitzenböck) for the representation of invariants of skew-symmetric tensors. Here, the results turn out to be more satisfactory. Symbolic expressions for the invariants of skew-symmetric tensors are more manageable and easier to compute than those for symmetric tensors. In fact, in contrast to symmetric tensors, the "meaning" of the vanishing of an invariant can be more easily gleaned from the symbolic representation, as we show by several examples.

In both instances, the actual invariant is obtained from the symbolic representation by applying an operator which we call the umbral operator. Invariants of symmetric tensors are obtained by applying the umbral operator to certain polynomials in a commutative algebra, whereas invariants of skew-symmetric tensors are obtained by applying the umbral operator to polynomials in an anticommutative algebra. Thus, the umbral operator can be viewed as mapping an anticommutative algebra into a commutative algebra, and vice versa. It is an instance of a Schur functor.

The umbral operator thus establishes a cryptomorphism between commutative and anticommutative algebras, and can be used to systematically develop anticommutative analogs of concepts of algebraic geometry. This, we believe, may ultimately turn out to be the main byproduct of the present investigation.

In the exposition, we have preferred to use algebras generated by an alphabet over algebras generated by a free module. The results can, however, be recast in a basis-free language: the choice between these two equivalent languages is largely a matter of taste and of the objectives at hand.

Synopsis

We prove two distinct but closely related results. The first is the extension of the standard basis theorem to superalgebras (defined below). The second is the application of the standard basis theorem to the computation of invariants (and, more generally, of covariants) of symmetric and skew-symmetric tensors. In this synopsis we give an informal description of the main ideas and results which can be read independently of the body of the work and which can be used as a guideline to the text.

We begin by recalling the three fundamental algebraic systems of invariant theory: the symmetric algebra, the divided powers algebra, and the exterior algebra.

(1) Given an alphabet A^0 (that is, a set A^0 whose elements are to be viewed as "variables"), the symmetric algebra $\mathrm{Symm}(A^0)$ generated by A^0 is the familiar commutative algebra of polynomials in the variables A^0. The coefficients of these polynomials will be integers, although (here and everywhere below) an arbitrary commutative ring with identity could be taken as the ring of coefficients.

(2) Given an alphabet A^-, the exterior algebra $\mathrm{Ext}(A^-)$ is the algebra generated by the variables A^-, subject to the identities $ab = -ba$ and $a^2 = 0$ for $a, b \in A^-$. Thus, $\mathrm{Ext}(A^-)$ is the algebra of "polynomials in anticommutative variables A^-." A nonzero monomial in $\mathrm{Ext}(A^-)$ is a product of a finite sequence of variables

$$a_1 a_2 \cdots a_n, \qquad a_i \in A^-,$$

where no two a_i coincide, and two monomials are related by the familiar "sign law"

$$a_1 a_2 \cdots a_n = (\operatorname{sgn} \sigma) a_{\sigma 1} a_{\sigma 2} \cdots a_{\sigma n}$$

for any permutation σ of the set $\{1, 2, \ldots, n\}$.

(3) Given an alphabet A^+, the divided powers algebra $\mathrm{Div}(A^+)$ is the commutative algebra generated by the variables $a^{(i)}$, as a ranges over A and as $i = 0, 1, 2, \ldots$. We set $a^{(0)} = 1$ and $a^{(1)} = a$. The idea is that $a^{(i)}$ is to satisfy the same identities as $a^i/i!$. Thus we impose the identities

$$(*) \qquad a^{(i)}a^{(j)} = \binom{i+j}{i} a^{(i+j)}$$

(other identities usually impose in the definition of the divided power algebra will not be needed, and need not be recalled here).

We wish to develop a suitable notation for the tensor product of these three algebras, for three disjoint alphabets A^0, A^-, and A^+. The usual notation of tensor products proves unwieldy, and we choose to describe the tensor product by a more direct route, namely, as the monoid algebra of a certain monoid to be presently defined. Thus, after taking the disjoint sum $A^0 \oplus A^- \oplus A^+ = A$ we consider a monoid $\mathrm{Mon}[A]$ which is "almost" the free monoid generated by A. The words in $\mathrm{Mon}[A]$ shall be products of variables in A^0 and in A^-, and of the divided powers $a^{(i)}$ for $a \in A$. Thus a word appears as a product, e.g.,

$$w = abc^{(i)}de^{(j)}.$$

The identities among these monomials are those that follow from the following commutation relations $ab = ba$ in all cases except when both a and b belong to A^-, in which case we set $ab = -ba$. Thus, if $a, b \in A^-$, if $c, d \in A^0$, and if $e, f \in A^+$ then we have, for example,

$$(**) \qquad dac^2e^{(3)}bf^{(5)} = -c^2bf^{(5)}ae^{(3)}d.$$

The product of two words is juxtaposition, except that products of divided powers are to be simplified by $(*)$. Thus, for example, $(ae)(be) = 2abe^{(2)}$. The length of a monomial is computed taking into account the fact that the divided power $a^{(i)}$ is to be considered of Length i. Thus, the monomial $(**)$ is of length (= degree) 13. With these conventions, the monoid algebra of $\mathrm{Mon}[A]$ is well defined by taking formal linear combinations and products. We call it the *superalgebra* $\mathrm{Super}[A]$ generated by the *signed alphabet* A. The variables of the alphabet A will be designated as neutral, negatively signed, and positively signed, respectively.

The superalgebra $\mathrm{Super}[A]$, although a worthwhile subject of investigation, is not the immediate object of the present study. We need to define a more complex structure, which will be called the *fourfold algebra.* To this end, we consider two signed alphabets $L = L^+ \oplus L^-$ and $P = P^+ \oplus P^-$, called *proper*, because neither has neutral elements. Their elements will be called *letters* and *places*, respectively. From these two alphabets we define a third signed alphabet $[L \mid P]$, the *letterplace alphabet.* The definition of $[L \mid P]$ is fundamental. The elements of $[L \mid P]$ will be pairs $(x \mid \alpha)$, where $x \in L$ and $\alpha \in P$; these pairs will sometimes be called letterplaces (the geometric motivation for this term will be given shortly). Their signatures are determined by the following rules:

(i) $(x \mid \alpha) \in [L \mid P]^+$ if $x \in L^+$ and $\alpha \in P^+$,

(ii) $(x \mid \alpha) \in [L \mid P]^0$ if $x \in L^-$ and $\alpha \in P^-$,

(iii) $(x \mid \alpha) \in [L \mid P]^-$ otherwise.

The Superalgebra Super$[L \mid P]$ is called the *fourfold algebra*. Our objective will be to show that the fourfold algebra is the suitable machinery to develop the invariant theory of symmetric and skew-symmetric tensors, and, as a byproduct, to formulate a "signed" generalization of linear algebra.

All theorems of linear and multilinear algebras can be viewed as consequences of the Laplace expansions for determinants. This sweeping assertion is in part made precise by the second fundamental theorem of invariant theory, and it will serve as motivation for the generalization of some such theorem to superalgebras that we shall develop below.

In ordinary linear algebra, one deals with vectors (here denoted by letters) and coordinates (also known as linear functionals, here denoted by places). The αth coordinate of a vector x is a *scalar* (here denoted by a neutral variable). Thus, vectors must be taken as negatively signed letters, and coordinates must be taken as negatively signed places. The value of a vector x at the coordinate α is denoted by $(x \mid \alpha)$ and is a scalar (that is, a neutral element). A *matrix* is a set $\{(x \mid \alpha); x \in X,\ \alpha \in A\}$ where X and A are respectively linearly ordered sets of letters and places of the same size, i.e., $|X| = |A| = n$. Relabeling the letters $x_1, \ldots, x_n$ and the places $\alpha_1, \ldots, \alpha_n$ by their linear order, the determinant of a matrix is then defined as usual (except for sign) as

$$(-1)^{n(n-1)/2} \sum_{\sigma} (\operatorname{sgn} \sigma)(x_1 \mid \alpha_{\sigma 1})(x_2 \mid \alpha_{\sigma 2}) \cdots (x_n \mid \alpha_{\sigma n}),$$

where σ ranges over all permutations of the set $\{1, 2, \ldots, n\}$ of indices. One verifies that the above expression equals (expansion by columns instead of rows)

$$(-1)^{n(n-1)/2} \sum_{\sigma} (\operatorname{sgn} \sigma)(x_{\sigma 1} \mid \alpha_1)(x_{\sigma 2} \mid \alpha_2) \cdots (x_{\sigma n} \mid \alpha_n).$$

We wish to recast this definition in a form that will be suitable for generalization to the fourfold algebra. To this end, we write the above determinant by the notation $(x_1 x_2 \cdots x_n \mid \alpha_1 \alpha_2 \cdots \alpha_n)$ and we note that this expression extends to a bilinear map from the exterior algebra Ext(X) in the letters, and from the exterior algebra Ext(A) in the places, with values in the symmetric algebra in the letterplaces $(x \mid \alpha)$. We are thus led to generalize such a bilinear map to superalgebras by suitably generalizing the notion of determinant and of the Laplace expansion that characterizes it. We first have to recast the classical Laplace expansions of a determinant in a language that is suitable for generalization. It turns out that the language of Hopf algebras is eminently suited to this purpose. We first review this language in the (classical) case of two exterior algebras $\text{Ext}(X)$ and $\text{Ext}(A)$ or, what is equivalent, to the case of the superalgebra $\text{Super}[L \mid P]$ where $L^0 = L^+ = P^0 = P^+ = \varnothing$. In this special case, the determinant can be viewed as a bilinear form from $\text{Super}[L] \times \text{Super}[P]$ to $\text{Super}[L \mid P]$. Let w be a word in $\text{Mon}[L]$ and let w' be a word in $\text{Mon}[P]$. We shall define $(w \mid w')$ as an element of $\text{Super}[L \mid P]$. The bilinear function $(w \mid w')$ shall satisfy the following conditions:

(i) $(w \mid w') = 0$ unless the words w and w' have the same length.

(ii) $(w \mid w') = (x \mid \alpha)$ if $w = x$ and $w' = \alpha$, that is, if the words are of length one, that is, if the words reduce to a single letter and place.

(iii) $(1 \mid 1) = 1$, where 1, the identity of the algebra, can be identified with a word of length zero.

Finally, we shall impose on $(w \mid w')$ the analog of Laplace expansions. To state these properly we recall that if w is a word in an exterior algebra, the *coproduct*

$$\Delta w = \sum_{w} w_{(1)} \otimes w_{(2)}$$

is the sum over all pairs of such words $w_{(1)}$ and $w_{(2)}$ such that $w_{(1)}w_{(2)} = w$, with suitable signs (incorporated in the notation) which need only be specified in the text.

In the notation of Hopf algebras, the Laplace expansion of a determinant $(w \mid w'w'')$ takes the pleasing form

$$\sum_{w} \pm (w_{(1)} \mid w')(w_{(2)} \mid w''),$$

where again we do not yet worry about signs. There is a similar expansion "by columns" that is similar to the present expansion by rows.

We now generalize this notation to an arbitrary fourfold algebra $\text{Super}[L \mid P]$. The fourfold algebra is the tensor product of four algebras, each one generated by those letterplaces $(x \mid \alpha)$ where x is either positive or negative, and α either positive or negative (recall that there are no neutral letters or places). Each factor in such a tensor product is an exterior algebra, a divided powers algebra, or a symmetric algebra.

Similarly, each of the superalgebras $\text{Super}[L]$ and $\text{Super}[P]$ is the tensor product of an exterior algebra and a divided powers algebra.

We now extend the definition of a "determinant" (which we call a *biproduct*) to the fourfold algebra as follows. If $w \in \text{Super}[L]$ and $w', w'' \in \text{Super}[P]$, we again define a bilinear form $(w \mid w')$ taking values in $\text{Super}[L \mid P]$ (note that we now allow L and P to be arbitrary proper alphabets, that is, alphabets without neutral elements). The biproduct $(w \mid w')$ will be subject to (i), (ii), and (iii) exactly as above, and to the technical condition

(iv) $(x^{(n)} \mid \alpha^{(n)}) = (x \mid \alpha)^{(n)}$ for divided powers of positively signed letters and places.

Finally, one assumes analogs of the Laplace expansions in the form

$$(*) \qquad (w \mid w'w'') = \sum_{w} \pm (w_{(1)} \mid w')(w_{(2)} \mid w''),$$

with suitable signs, where

$$\Delta w = \sum_{w} w_{(1)} \otimes w_{(2)}$$

is the coproduct in the Hopf algebra Super[L] (see below for further explanations). One also assumes a similar Laplace expansion where the roles of letters and places are interchanged.

The sum on the right of the Laplace expansion (*), as well as the sign of each term, can be understood without knowledge of Hopf algebras as follows. The sum on the right of (*) ranges over all distinct ordered pairs $w_{(1)}$, $w_{(2)}$ of subwords of w such that $w_{(1)}w_{(2)} = w$. By (i), the only nonzero terms are given by pairs $w_{(1)}, w_{(2)}$ such that Length$(w_{(2)})$ = Length(w') and Length$(w_{(2)})$ = Length(w''). The sign of each term on the right of (*) is determined by the following rule. In the monoid Mon[$L \oplus P$] generated by the disjoint sum of the alphabets L and P, consider the words $ww'w''$ and $w_{(1)}w'w_{(2)}w''$. Evidently the second word can be obtained from the first by a succession of transpositions of adjacent elements of $L \oplus P$ (with proper care given to divided powers). The sign of the corresponding term on the right of (*) is then the parity of the number of such transpositions, for which the two adjacent elements of $L \oplus P$, which are being transposed, are both negatively signed. We stress the fact that of the two letters being transposed, one may belong to L and another to P.

EXAMPLE. Suppose all letters are positively signed, and places are of arbitrary signature. Then

$$(a^{(2)}b \mid \alpha\beta\gamma) = (a^{(2)} \mid \alpha\beta)(b \mid \gamma) + (ab \mid \alpha\beta)(a \mid \gamma).$$

(v) Finally, one assumes a dual Laplace expansion

$$(w'w'' \mid w) = \sum \pm (w' \mid w_{(1)})(w'' \mid w_{(2)})$$

with a similar rule for the signs.

We show in the text that these expansion rules are consistent. They define a bilinear form, which we call the *biproduct*, which can be viewed as a signed generalization of the determinant. When w and w' are products of distinct positively signed letters and places, then the biproduct $(w \mid w')$ reduces to the permanent. This is not the case, however, for $(w \mid w')$ when w and w' are arbitrary products of divided powers. In this case, the pair $(w \mid w')$ can in no way be seen as the generalization of the determinant or permanent of a matrix (unless one is willing to consider the possibility that the set of rows and the set of columns of such a "matrix" shall be allowed to become multisets).

The biproduct preserves all formal properties (that is, the Laplace expansions) of a determinant, except for signs. As already remarked, all of linear and multilinear algebra can be recast in terms of Laplace identities. It stands to reason, therefore, to expect that the biproduct will yield a generalization of linear algebra to "signed" vector spaces and signed modules generally. In this paper we take two steps toward the realization of this program, which we now proceed to informally describe.

The first step is the extension to the fourfold algebra of the standard basis theorem of Doubilet, Rota, and Stein (a special case of our result, which in the present notation corresponds to setting $L = L^-$ and $P = P^-$). Such a generalization proceeds as follows. Define a Young diagram over L (and similarly over P) as a sequence $D = (w_1, w_2, \ldots, w_n)$ of words w_i in Mon$[L]$, such that $\lambda_i = \text{Length}(w_i)$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The vector $\lambda = (\lambda_1, \lambda_2, \ldots)$ is the *shape* of the Young diagram. If

$$w_i = x_{i1}x_{i2}\cdots x_{i\lambda i}, \qquad x_{ij} \in L,$$

where we have written out each divided power by repeating the same letter (strictly speaking, this is an abuse of notation), then the words

$$\tilde{w}_j = x_{ij}x_{2j}\cdots x_{\lambda j \cdot j}$$

define the *dual diagram* $\tilde{D}$ of shape $\tilde{\lambda}$.

A Young diagram D over L (or over P) will be said to be *standard* when, for every pair of words

$$w_i = x_1x_2\cdots x_r, \qquad x_j \in L,$$
$$\tilde{w}_i = y_1y_2\cdots y_s, \qquad y_j \in L,$$

where w_i is in D and $\tilde{w}_i$ is in $\tilde{D}$ the following conditions are satisfied. Let us refer to w_i as a *row* of the diagram and to $\tilde{w}_i$ as a *column* of the diagram. Choose a linear order on the alphabet L, which will remain fixed from now on. Then

(i) two successive letters x_i and x_{i+1} in the same row are in nondecreasing order ($x_i \leq x_{i+1}$) if they are both positively signed and in strictly increasing order ($x_i < x_{i+1}$) if they are both negatively signed.

(ii) two successive letters y_i and y_{i+1} in the same column of the tableau D (i.e., in the same row of the tableau $\tilde{D}$) are in strictly increasing order ($y_i < y_{i+1}$) if they are both positively signed and in nondecreasing order ($y_i \leq y_{i+1}$) if they are both negatively signed.

(iii) two successive letters in the same row (or in the same column) which are of different signatures are in strictly increasing order.

A similar definition is given for Young diagrams made out of P.

Now let D and $E = [w'_1, \ldots, w'_n]$ be Young diagrams of the same shape made out of the alphabets L and P, respectively. We define the *Young tableau* of the diagram pair D, E to be

$$\text{Tab}(D \mid E) = \pm(w_1 \mid w'_1)(w_2 \mid w'_2)\cdots(w_n \mid w'_n).$$

(This is a slight oversimplification, because of the possible occurrence of divided powers—the text gives the precise definition.)

A *standard Young tableau* $\mathrm{Tab}(D \mid E)$ is a Young tableau where both diagrams D and E are standard. Our main result states that standard Young tableaux are an integral basis of the superalgebra $\mathrm{Super}[L \mid P]$. Actually, the final result is stronger. It states that a (not necessarily standard) tableau $\mathrm{Tab}(D \mid E)$ is uniquely expressible as a linear combination with integer coefficients of standard Young tableaux. If $\mathrm{Tab}(D \mid E)$ is of shape λ, then the only standard Young tableaux occurring with nonzero coefficients are of shapes which are greater than λ in the dominance order of shapes. Thus the submodules, spanned by all tableaux whose shapes form an order ideal in the partially ordered set of shapes under the dominance order, span an invariant submodule of $\mathrm{Super}[L \mid P]$, under permutations of both letters and places.

An immediate corollary of this theorem is that the number of standard Young diagrams is independent of the linear ordering chosen for the alphabet. No direct combinatorial proof of this surprising fact is known at present.

The proof of the standard basis theorem proceeds in two steps. The fact that the standard tableaux span the superalgebra is a consequence of the following *straightening formula*

$$\sum_{w} \pm(vw_{(1)} \mid u')(w_{(2)}w \mid u) = \sum_{u,v} \pm(wv_{(1)} \mid u'u_{(1)})(v_{(2)}w' \mid u_{(2)}).$$

(For the actual computation of the signs, consult the text.)

The straightening formula leads to a recursive algorithm (ultimately going back to Alfred Young) to express any monomial in the superalgebra as an integral linear combination of standard tableaux.

The fact that the standard tableaux are linearly independent is subtler; it is based on a refined duality. One defines a dual alphabet $L^* = L^{*+} \cup L^{*-}$ with L^{*+} in bijective correspondence with L^- and L^{*-} with L^+. Thus, if x is a letter, then x^* is a letter of the opposite sign; similarly, for P^*. One then defines a scalar bilinear form

$$\langle p, q \rangle, \qquad p \in \mathrm{Super}[L^* \mid P^*],\ q \in \mathrm{Super}[L \mid P],$$

by the following rules:

(i) $\langle (x^* \mid \alpha^*), (y \mid \beta) \rangle = 0$ if $x \neq y$ or $\alpha \neq \beta$, and

(ii) $\langle (x^* \mid \alpha^*), (x \mid \alpha) \rangle = \pm 1$, where the sign on the right is negative if and only if both x and α^* are negatively signed. To see the motivation for this rule, we (quite unrigorously) rewrite the left side as

$$(x^* \mid \alpha^*)(x \mid \alpha) = \pm (x^* \mid x)(\alpha^* \mid \alpha),$$

and we see that the minus sign is due to the "fact" that the elements x and α^* have been commuted.

(iii) Analogs of Laplace expansions. For example, if $w'w'' \in \mathrm{Super}[L^* \mid P^*]$ and $w \in \mathrm{Super}[L \mid P]$, we set

$$\langle w'w'', w \rangle = \sum_{w} \pm \langle w', w_{(1)} \rangle \langle w'', w_{(2)} \rangle.$$

Here, the coproduct

$$\Delta w = \sum_w w_{(1)} \otimes w_{(2)}$$

is taken in the Hopf algebra Super$[L \mid P]$, which is the tensor product of two Hopf algebras; that is, the pairs $w_{(1)}$ and $w_{(2)}$ range over all subwords of w, for $w \in$ Super$[L \mid P]$, such that $w = w_{(1)}w_{(2)}$. Again, the signs on the right are determined according to the number of transpositions of adjacent negatively signed elements (letters or places).

The byproduct of this seemingly unusual setup for duality of two algebras is the generalization to superalgebras of the celebrated result of A. Young (in Young's language, stating that $P(12)N(12) = 0$). In this present context, we prove that $\langle (w^* \mid u^*), (w' \mid u') \rangle = 0$ whenever the length of the biproducts is two or more. This leads to a considerable simplification of the computation of

$$\langle \text{Tab}(D^* \mid E^*),\ \text{Tab}(D' \mid E') \rangle.$$

One finds upon evaluating that on the right side only sums of certain simple monomials appear (which we have called Gale-Ryser interpolants after a famous theorem of Gale and Ryser on the existence of 0-1-matrices having given row and column sums). One goes on to establish a triangular biorthogonality property for standard tableaux, from which linear independence is inferred.

In order to describe the invariant-theoretic results, we shall assume from now on that $P = P^- = \{1, 2, \ldots, n\}$. We define a *bracket* (of length n) to be the element $[w] = (w \mid 1\,2 \cdots n)$ of Super$[L \mid P]$. Clearly the bracket will vanish unless the word w in Super$[L]$ is of length n. From now on we shall separately consider either of the two special cases $L = L^-$ or $L = L^+$. In the case $L = L^-$, the bracket coincides with the classical bracket first defined by Cayley; that is, if $w = x_1 x_2 \cdots x_n$, the bracket is the determinant of the matrix of n row vectors x_i which have the entries

$$x_i = ((x_i \mid 1), (x_i \mid 2), \ldots, (x_i \mid n)).$$

Note that because $L = L^-$ and $P = P^-$, each entry is a neutral element, and thus can legitimately be called a scalar. The straightening formula now specializes to give identities on brackets of the form

$$(**) \qquad \sum_w \pm [v w_{(1)}][w_{(2)} w'] = \sum_v \pm [w v_{(1)}][v_{(2)} w'],$$

which, as is known, can be used for an abstract characterization of the bracket. All of linear algebra is "coded" in these identities.

In the second case, $L = L^+$, one obtains a remarkable generalization of the determinant. The bracket $[w]$, for $w \in$ Super$[L]$ = Super$[L^+]$, will now no longer be scalar valued but will take its value in an exterior algebra. We shall call it the *skew bracket*. It can no longer be viewed as the determinant or the permanent of a "matrix", unless w is a product of n distinct letters. One computes, for example, $[a^{(n)}] = (a \mid 1)(a \mid 2)\cdots(a \mid n)$, where each $(a \mid i)$ is an element of degree one of an exterior algebra. More generally, one has

$$[w][w'] = (-1)^n [w'][w],$$

and identities of the form (**) hold for skew brackets (with fewer signs) and in fact can be used to characterize skew brackets. Thus, the possibility arises that skew brackets are the syntax for a semantic construct which will be a "skew" analog of linear algebra. We shall leave this enticing possibility untouched for the moment, and proceed directly to the invariant-theoretic applications of the bracket (both classical and skew).

For ease of exposition, we deal separately with symmetric and skew-symmetric tensors, keeping the narrative as informal as possible, at the cost of some inaccuracies which are corrected in the text.

Given a vector space V of dimension n over a field of characteristic zero, a symmetric tensor t over V (that is, a homogeneous element of the symmetric algebra $S(V)$) has, relative to a basis $e_1, e_2, \ldots, e_n$, the coordinates $a_{j_1 j_2 \cdots j_n}$, so that

$$t = \sum \binom{k}{j_1, j_2, \ldots, j_n} a_{j_1 j_2 \cdots j_n} e_1^{j_1} e_2^{j_2} \cdots e_n^{j_n},$$

where the sum ranges over all n-tuples $(j_1 \cdots j_n)$; note that almost all the terms equal zero (because the multinomial coefficient vanishes). The integer k is the degree or *step* of the tensor. An *invariant* I of t is a polynomial in the coordinates $a_{j_1 j_2 \cdots j_n}$ which is independent of the choice of the basis, except for a constant factor. Thus, if I is an invariant, then the *condition* $I = 0$ is *geometric*; that is, it is independent of the coordinate system. The idea of invariant theory is to express by the vanishing of invariants certain logically or combinatorially defined properties of tensors such as decomposability, rank, divisibility by another tensor, etc. To this end, a slight extension of the notion of invariant is needed, namely, the notion of a covariant.

Observe that a *joint invariant* of a set of tensors may be defined in the obvious way. When the coordinates of one or more of the tensors of such a set are allowed to be independent transcendentals (adjoined to the base field), a joint invariant will turn out to be a polynomial in these independent transcendentals. Such a polynomial is called a *covariant*, with the understanding that a covariant vanishes when it vanishes identically as a polynomial in its independent transcendentals.

We next describe the symbolic method for the representation of all invariants (and hence all covariants) of symmetric tensors. To every (symmetric) tensor t we associate an indefinite supply of letters (or *symbols*) belonging to a *negatively signed* alphabet L^-, and we say that a letter "belongs" to the tensor t.

We next define an operator U, the *umbral operator*, from the superalgebra Super$[L \mid P]$ to the "scalar" coordinates of a tensor of step k. We simply set

$$\langle U, (a \mid 1)^{j_1} (a \mid 2)^{j_2} \cdots (a \mid n)^{j_n} \rangle = a_{j_1 j_2 \cdots j_n},$$

where a is any of the letters of L belonging to a tensor t of step k, and where $j_1 + j_2 + \cdots + j_n = k$. If $j_1 + j_2 + \cdots + j_n \neq k$, the right side is set equal to zero. Note that all $(a \mid i)$ are neutral elements.

If w is any monomial in Super$[L \mid P]$ (a symmetric algebra) we can write $w = w(a)w(b)\cdots$ when $w(a)$ is the product of all letterplaces $(a \mid i)$ containing the letter a, etc. We set

$$\langle U, w \rangle = \langle U, w(a)w(b)\cdots \rangle = \langle U, w(a) \rangle \langle U, w(b) \rangle \cdots .$$

Finally, we extend the definition of U to all of Super$[L \mid P]$ by linearity. The main result states that a polynomial I is an invariant of a set of symmetric tensors if and only if $I = \langle U, p \rangle$, where p is a polynomial in brackets (in the present case of symmetric tensors, the brackets are nothing but ordinary determinants).

The classification of invariants can be further simplified by applying the standard basis theorem. According to this theorem, any polynomial in brackets can be integrally expressed as an integral linear combination of bracket products which correspond to standard tableaux, in the given linear ordering of the alphabet L. Thus, it suffices (by and large) to consider the case where p is a standard tableau in Super$[L \mid P]$, where each row is of length n (thus, the n places $1, 2, \ldots, n$ appear in each row on the right side of the tableau). If x_i is a transcendental, we can safely set

$$\langle U, (x \mid i)^j \rangle = x_i^j,$$

thereby identifying x with a symbol. If we further agree that the linear order of L shall be chosen in such a way as to place last all symbols corresponding to transcendentals (that is, after all symbols corresponding to genuine tensors), then a standard tableau for a covariant is partitioned into an upper part, containing all symbols corresponding to the tensor(s), and a lower part, containing only (symbols for) transcendentals. For all practical purposes, only the upper part will matter in the description of the covariant. We thus succeed in associating to every covariant a standard Young diagram in the familiar form such as

$$\begin{matrix} abcd \\ abe \\ bfg. \end{matrix}$$

The invariant theory of symmetric tensors can be carried much farther by the symbolic method, but the present exposition now turns to the much less explored subject of skew-symmetric tensors, where previous work (done largely in the nineteenth century) was considerably more limited in scope. Our objectives will be, first, to develop a symbolic method that will be strikingly analogous to the symbolic method for symmetric tensors, and second, to show by examples that the method is (for skew-symmetric tensors better than for symmetric tensors) effective both for the construction and for the interpretation of invariants and covariants.

The method is "dual" to the one described above for symmetric tensors, in the sense that the words "commutative" and "anticommutative" are judiciously interchanged. It is not clear a priori, however, how such an interchange should be carried out, and in fact the method we are about to describe reveals new clues to how this interchange may be effective in disparate situations.

A skew-symmetric tensor t of step k is a homogeneous element of the exterior algebra $\bigwedge(V)$. Relative to a basis $e_1, e_2, \ldots, e_n$ of V, the tensor can be written in the form

$$t = \sum_{i_1 < i_2 < \cdots < i_k} (a_{i_1 i_2 \cdots i_k}) e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$$

where the sum ranges over all subsets $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$, with the customary conventions relative to signs, namely.

$$a_{\sigma_1 \sigma_2 \cdots \sigma_k} = (\operatorname{sgn} \sigma) a_{12 \cdots k}.$$

An invariant I of t is a polynomial in the coordinates $a_{i_1 i_2 \cdots i_k}$ which is independent of the coordinates, except for a constant factor, so that (again) the condition $I = 0$ is geometric. Joint invariants and covariants are defined as in the symmetric case.

We now describe the delicate symbolic method by which all invariants (and covariants) of skew-symmetric tensors can be represented. Choose a linearly ordered alphabet L which is now *positively signed* ($L = L^+$). Recall that the place alphabet P is the *negatively signed* set $\{1, 2, \ldots, n\}$. To every (skew-symmetric) tensor t we assign an infinite supply of letters "belonging" to it, and we assume that the supply is ample enough for all purposes. In addition, we assume that L contains an indefinitely large supply of letters belonging to an indefinitely large supply of independent transcendentals that may be adjoined to the ground field if and when necessary. We next define an umbral operator U that (again) maps the superalgebra Super$[L \mid P]$ to the field of scalars, eventually with transcendentals adjoined.

If a is a letter belonging to the skew-symmetric tensor t of step k, we set

$$\langle U, (a^{(k)} \mid i_1 i_2 \cdots i_k) \rangle = a_{i_1 i_2 \cdots i_k}.$$

On the right side is the scalar component of the tensor t relative to the basis $e_1, e_2, \ldots, e_n$, which will remain fixed from now on. On the left side, $a^{(k)}$ is the kth divided power of the letter a, so that (by the properties of the byproduct)

$$(a^{(k)} \mid i_1 i_2 \cdots i_k) = (a \mid i_1)(a \mid i_2) \cdots (a \mid i_k).$$

Note that for $i \neq j$ we have $(a \mid i)(a \mid j) = -(a \mid j)(a \mid i)$, since $(a \mid i)$ is negatively signed to all $a \in L$ and $i \in P$.

Next, set $\langle U, (a^{(j)} \mid i_1 i_2 \cdots i_j) \rangle = 0$, if $j \neq k$.

Now let n be an arbitrary monomial in Super$[L \mid P]$. By altering the sign of m, if necessary, we can write

$$m = \pm m(a) m(b) \cdots,$$

where $m(a)$ is the product of all letter places $(a \mid i)$ containing the letter a, etc., and, what is essential, where $a < b < \cdots$ in the linear order of L. Thus,

$$m(a) = (a^{(j)} \mid i_1 i_2 \cdots i_j)$$

for some j, similarly for $m(b)$, etc.

Now set

$$\langle U, m(a)m(b)\cdots\rangle = \langle U, m(a)\rangle\langle U, m(b)\rangle \cdots ,$$

and extend to Super$[L \mid P]$ by linearity. We prove that the operator U is well defined. For example, let a and b be both letters belonging to the tensor t, and suppose that $a < b$. Then

$$\langle U, (a^{(k)} \mid 12\cdots k)(b^{(k)} \mid 12\cdots k)\rangle = (a_{12\cdots k})^2,$$

but

$$\langle U, (b^{(k)} \mid 12\cdots k)(a^{(k)} \mid 12\cdots k)\rangle = -(a_{12}\cdots k)^2.$$

if k is odd. As another example, let u and v be vectors in V, that is, skew-symmetric tensors of step one. Then we have

$$\langle U, (a \mid i)\rangle = u_i, \qquad \langle U, (b \mid i)\rangle = v_i,$$

if a belongs to u and b belongs to v. Thus, if $a < b$, then

$$\langle U, (ab \mid ij)\rangle = \langle U, (a \mid i)(b \mid j) + (b \mid i)(a \mid j)\rangle = u_i v_j - u_j v_i,$$

and we find (spectacularly enough) that

$$\langle U, (ab \mid ij)\rangle = \langle U, (ba \mid ij)\rangle,$$

as we might well expect, since $(ab \mid ij) = (ba \mid ij)$ in Super$[L \mid P]$, and since U is well defined.

More generally, let $u^1, u^2, \ldots, u^n$ be vectors, and let $a_1 < a_2 < \cdots < a_n$ be symbols (letters) belonging to each. Then we find

$$\langle U, [a_1 a_2 \cdots a_n]\rangle = [u^1 u^2 \cdots u^n]$$

where the left bracket is the bracket in Super$[L \mid P]$, and the right bracket is the ordinary determinant of the vectors $u^1, \ldots, u^n$ relative to the basis $e_1, \ldots, e_n$. In particular, we find that

$$\langle U, [a_{\sigma 1} a_{\sigma 2} \cdots a_{\sigma n}]\rangle = [u^1 u^2 \cdots u^n]$$

for any permutation σ. This point is worth stressing, because it is the point at which Weitzenböck's attempt to develop a symbolic method for skew-symmetric tensors failed. Weitzenböck (like all other classical invariant theorists) failed to distinguish between a vector u and a symbol a representing u, and used the same notation for both. As a consequence, his brackets $[u^1 u^2 \cdots u^n]$ were to be taken sometimes as symmetric and sometimes as skew-symmetric relative to permutations of $\{1, 2, \ldots, n\}$ depending on the context! Small wonder that few invariants (other than those previously known) should have been computed by such a technique.

Our main result is formally similar to the main theorem for symmetric tensors. It states that every invariant (or joint invariant) of a skew-symmetric tensor t can be written in the form $\langle U, p\rangle$, where p is a polynomial in skew brackets containing only symbols belonging to t.

Again, the listing of invariants is simplified by applying the standard basis theorem to the subalgebra of Super$[L \mid P]$ spanned by polynomials in skew brackets. A skew bracket monomial, that is, a product of skew brackets

$$m = [w_1][w_2]\cdots[w_j],$$

where w_i is a word in Super$[L]$ which is of Length n, stands for a Young diagram $D = (w_1, w_2, \ldots, w_j)$. Let us write $n = \text{Tab}(D)$. (The tableaux in the places need not be written out, since they all consist of successions of the single word $w' = 12\cdots n$.) By the standard basis theorem, those monomials $m = \text{Tab}(D)$, where D is a standard Young diagram, span the subspace of Super$[L \mid P]$ of bracket polynomials. Since $L = L^+$, a diagram D will be standard if along each row the letters appear in nondecreasing order, and along each column the letters appear in strictly increasing order. In other words, two successive letters a and b in a word w_i must satisfy the condition $a \le b$, whereas the jth letters c and d of two successive words w_i and w_{i+1} must satisfy the condition $c < d$.

For covariants, it is again convenient to let the letters belonging to vectors (and tensors) with independent transcendental entries be last in the order of the alphabet L, so that D partitions into an upper and a lower part as before. The upper part of D suffices to define a covariant. In other words, a covariant is to be specified by a Young diagram $D' = (w_1', \ldots, w_j')$ where Length$(w_i') \le n$. The covariant will be $\text{Tab}(w_1'w_1'', \ldots, w_j'w_j'')$, where $w_i'w_i''$ is of length n, and where the words w_i'' are made up of symbols (letters) belonging to independent transcendentals.

Contrary to what happens for symmetric tensors, the translation of a geometric property into the vanishing of covariants turn out to be successful for skew-symmetric tensors, and we devote Chapter 5 to the discussion of a number of such examples old and new.

For a tensor t of step 2, or bivector, the only invariant is easily proved to be the Pfaffian, which for $n = 2k$ is symbolically represented by

$$a_1^{(2)} a_2^{(2)} \cdots a_k^{(2)},$$

where a_i are distinct letters belonging to t. The covariants of t are represented by incomplete Pfaffian

$$a_1^{(2)} a_2^{(2)} \cdots a_j^{(2)}, \qquad j \le k,$$

and (as is well known) they give the rank of the tensor.

For a tensor of step k, the covariants

$$C_p: \begin{matrix} a_1 a_2 \cdots a_p \\ a_1^{(k-1)} \\ \vdots \\ a_p^{(k-1)} \end{matrix}$$

determine the rank of the tensor: if C_{r+1} vanishes but C_r does not vanish, then the tensor is of rank r.

The classical Grassmann conditions for a tensor to be decomposable translate into the vanishing of either of the two covariants

$$\begin{matrix} a^{(k)} b \\ b^{(k-1)} \end{matrix} \quad \text{or} \quad \begin{matrix} a^{(k)} b^{(2)} \\ b^{(k-2)} \end{matrix}$$

where a and b are distinct letters belonging to the tensor.

The condition that a tensor of step 3 be divisible by a vector is the vanishing of the covariant

$$\begin{matrix} a^{(2)} b^{(3)} \\ a c^{(3)} \end{matrix}$$

where a, b, c are symbols belonging to the same tensor. These conditions are generalized in the text to give covariants specifying whether a given tensor of arbitrary step is divisible by one, two, etc., linearly independent vectors.

In conclusion, we list those covariants which specify each of the canonical forms for a tensor of step 3 in dimension $n = 6$ and 7.

For $n = 6$ the covariants that specify each of the four canonical forms are

$$\begin{matrix} a^{(3)} b^{(2)} \\ b; \end{matrix}$$

$$\begin{matrix} a^{(3)} b^{(2)} c \\ b c^{(2)}; \end{matrix}$$

and

$$\begin{matrix} a^{(3)} b^{(2)} c \\ b c^{(2)} d^{(3)}. \end{matrix}$$

We find it remarkable that each of these turns out to be a standard Young diagram. A similar phenomenon happens for $n = 7$.

Notation Guide

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Dedicated
to the memory
of
Alfredo Capelli,
H. W. Turnbull,
Alfred Young,
Roland Weitzenböck.

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