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Number 72

Introduction to Arrangements

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to my parents

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Preface

An arrangement of hyperplanes is a finite collection of codimension one subspaces in a finite dimensional vector space over some field. Arrangements occur in several branches of mathematics: in the study of braids and phase transition, in wave fronts, in hypergeometric functions, in reflection groups and Lie algebras, in coding theory, in the study of certain singularities, in combinatorics and group theory, and in spline functions.

Some aspects of the theory have a distinguished history. This was reviewed in Grünbaum's book [60] and in his CBMS lectures [62]. Recent interest in the topological properties of the complement of an arrangement over the complex numbers started with papers by Arnold [3], Brieskorn [20], Deligne [33] and Hattori [64]. They studied the cohomology groups and the homotopy type of the complement. Orlik and Solomon [100] added combinatorial tools and Terao [139] used methods of algebraic geometry. These results were described by Cartier [24] in a Bourbaki seminar talk. These lecture notes provide an introduction to the new developments and survey the current activity in the area, with particular emphasis on the topological aspects. A more comprehensive treatment is forthcoming in a book written jointly with Louis Solomon and Hiroaki Terao [111].

I have received financial support from the National Science Foundation, the Wisconsin Alumni Research Foundation, the Mathematical Sciences Research Institute, Berkeley, and the Japan Society for the Promotion of Science. Parts of these notes were written at MSRI, Berkeley and at RIMS, Kyoto.

During the preparation of these lectures I visited several universities. I would like to thank my hosts for their hospitality: Eiichi Bannai in Columbus, Per Holm in Oslo, Haakon Waadeland in Trondheim, Michel Kervaire in Geneva, Rob Kirby and Emery Thomas in Berkeley, Kyoji Saito in Kyoto, and Mutsuo Oka in Tokyo.

Mike Falk's idea to organize this meeting gave the impetus to write these notes. He also helped me understand the work on minimal models. The presentation of the topological part owes a great deal to his PhD thesis [40], which was the first careful exposition of the foundational material. Arrangements are studied extensively by Soviet mathematicians. I am grateful to V. I. Arnold for references to this work. Louis Solomon and Hiroaki Terao taught me much of the contents of these notes and gave me permission to use material from our forthcoming book. I owe them special thanks.

Finally, I want to thank the participants of the conference in general, and Curtis Greene, Dick Randell, Tom Zaslavsky, and Sergey Yuzvinsky in particular, for their interest, enthusiasm, and help. The present version of the notes incorporates their suggestions for changes and corrections of the preliminary text distributed at the meeting.

Madison, October 23, 1988

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