Klaus Schmidt

ALGEBRAIC IDEAS IN ERGODIC THEORY
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factors, transformations.

\text{Math Type endomorphisms} the normalizers \( C^* \), and sequences, structure.

 flows of weights \( d \).

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Notation

\[\alpha^N \] -action defined by a module \( N \)
\( \mathcal{A}(R), \mathcal{A}'(R) \) von Neumann algebras of a nonsingular equivalence relation
\( B^1(R, A) \) set of coboundaries of \( R \) with coefficients in a group \( A \)
\( B(H) \) = the set of bounded linear operators on a Hilbert space \( H \)
\( \beta_P \) beta-function of a Markov shift
\( \mathbb{C} \) = the complex numbers
\( c_B \) restriction of a cocycle \( c \) to a set \( B \)
\( c_p \) generator of \( \Gamma_p/\Delta_p \)
\( \mathcal{E}(R) \) a \( C^* \)-algebra of an equivalence relation \( R \)
\( \delta_{a, b} \) = the Kronecker delta (\( \delta_{a, b} = 1 \) if \( a = b \), and \( \delta_{a, b} = 0 \) otherwise)
\( \Delta_p \) a group associated with a nonnegative matrix \( P \)
\( (\mathcal{D}(R), \mathcal{D}(R)^+) \) dimension module
\( \mathcal{E}(c), \mathcal{E}^-(c) \) essential range of a cocycle \( c \)
\( GL(n, \mathbb{Z}) \) = the group of invertible \( n \times n \) matrices with integer entries
\( \Gamma_p \) a group associated with a nonnegative matrix \( P \)
in an abelian group \( A \)
\( h(\cdot) \) entropy
\( H^1(R, A) \) first cohomology group of \( R \) with coefficients
in an abelian group \( A \)
\( \mathcal{J}_R : S \) index cocycle of a subrelation
\( k_\pi \) = algebraic closure of the prime field \( F_\pi = \mathbb{Z}_\pi \)
of characteristic \( \pi \)
\( K_0(\mathcal{A}) \) dimension group of a \( C^* \)-algebra
\( L_V \) nonsingular automorphism of \( R \) defined
by an automorphism \( V \in [R] \)
\( L_V, L'_V \) operators in \( \mathcal{A}(R) \) and \( \mathcal{A}(R) \) associated
with an automorphism \( V \in [R] \)
\( \lambda(V) \) module of \( V \in \mathcal{N}(R) \)
\( \mu_B \) = restriction of a measure \( \mu \) to a measurable
set \( B \) with \( \mu(B) > 0 \)
\( m_P \) measure of maximal entropy on a Markov shift
\( \mu_P \) Markov measure on a two-sided Markov shift \( X_P \)
\( \mu_R, \mu_R^{(L)}, \mu_R^{(R)} \) measures on a nonsingular equivalence relation \( R \)
\( \mathcal{M}(R), \mathcal{M}(R) \) algebra of multiplication operators in \( \mathcal{A}(R) \)
and \( \mathcal{A}(R) \)
\( \mathbb{N} \) = \( \{0, 1, 2, \ldots\} \)
\( \mathbb{N}^\times \) = \( \{1, 2, 3, \ldots\} \)
\[ \nu_p \]
Markov measure on a one-sided Markov shift \( Y_p \)

\[ \mathcal{N}(\mathbb{R}) \]
normalizer of an equivalence relation \( \mathbb{R} \)

\[ P, P \]
nonnegative, irreducible matrix and the associated
stochastic matrix

\[ \pi(V) \]
outer period of \( V \in \mathcal{N}(\mathbb{R}) \)

\[ \mathbb{Q} \]
the rationals

\[ \mathbb{R} \]
the real numbers

\[ \mathbb{R}_+ \]
\([0, \infty) \subset \mathbb{R} \)

\[ \mathbb{R}_+ \times \]
\( = \mathbb{R} \times \{0\} \)

\[ \mathbb{R}_+ \]
\( = \mathbb{R}_+ \cap \mathbb{R}_+ \times \)

\[ \Re(\alpha) \]
real part of a complex number \( \alpha \)

\[ [\mathbb{R}] \]
full group of an equivalence relation

\[ [[\mathbb{R}]] \]
ample group

\[ \mathbb{R}_B \]
equivalence relation induced on \( B \)

\[ r(c) \]
cohomology invariant

\[ \mathbb{R}(c) \]
skew product relation defined by a cocycle \( c \)

\[ \mathbb{R}(c, \mathbb{R}) \]
subrelation defined by a cocycle

\[ \mathcal{R}_d \]
ring of Laurent polynomials

\[ \mathbb{R}^P, \mathbb{R}^{P_P} \]
nonsingular equivalence relations on Markov shifts

\[ [\mathbb{R}] : \mathbb{S} \]
index of a subrelation

\[ \mathbb{R}^T \]
equivalence relation of a nonsingular group action \( T \)

\[ \mathbb{R}^V \]
equivalence relation of a nonsingular automorphism or
endomorphism \( V \)

\[ \mathbb{R}^V \]
nonsingular automorphism of \( \mathbb{R} \) associated with an
automorphism \( V \in [\mathbb{R}] \)

\[ \rho_{\mathbb{R}, \mu} \]
Radon–Nikodym derivative of a nonsingular relation \( R \)

\[ |S| \]
CARDINALITY OF A SET \( S \)

\[ S_1 \]
\( = \{ z \in \mathbb{C} : |z| = 1 \} \)

\[ \mathcal{S}_B \]
\( = \sigma \)-algebra \( \mathcal{S} \) induced on a set \( B \)

\[ \sigma(\mathfrak{F}, P) \]
higher dimensional Markov shift

\[ \text{SL}(n, \mathbb{Z}) \]
the group of \( n \times n \) matrices with integer entries and
determinant 1

\[ S^P, S^{P_P} \]
nonsingular equivalence relations on Markov shifts

\[ \sigma_P \]
Markov shift

\[ \mathcal{S}^T \]
\( = \sigma \)-algebra of \( T \)-invariant subsets in \( \mathcal{S} \), where \( T \)
is a group action

\[ S(f) \]
the support of a polynomial \( f \)

\[ S^V \]
equivalence relation of a nonsingular endomorphism \( V \)

\[ T \]
\( = \mathbb{R}/\mathbb{Z} \)

\[ T^{(c)} \]
group action on a skew product

\[ T^{(c)} \]
skew product action

\[ \mathcal{F}(c) \]
\( \mathcal{F} \)-set of a cocycle

\[ \mathcal{W}^u(x), \mathcal{W}^s(x) \]
stable and unstable sets of a point \( x \) in a two-sided
Markov shift

\[ \mathcal{X}_{(\mathcal{F}, P)} \]
higher dimensional Markov shift space

\[ (\mathcal{X}^V, \alpha^V) \]
dynamical system associated with a module

\[ \mathcal{X}_P \]
two-sided Markov shift space associated with a nonnegative
matrix \( P \)

\[ \mathcal{X}_P^* \]
set of doubly transitive points in \( \mathcal{X}_P \)
\( Y_P \) one-sided Markov shift space associated with a nonnegative matrix \( P \) 8
\( \mathbb{Z} \) = the integers
\( \mathbb{Z}/n\mathbb{Z} \)
\( \mathbb{Z}^1(\mathbb{R}, A) \) set of (1-)cocycles of \( \mathbb{R} \) with coefficients in a group \( A \) 18
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