

conference board of the mathematical sciences  
regional conference series in mathematics

number **76**



**Klaus Schmidt**

**ALGEBRAIC IDEAS  
IN ERGODIC THEORY**



supported by the national science foundation  
published by the american mathematical society

**ALGEBRAIC IDEAS  
IN ERGODIC THEORY**

*This page intentionally left blank*

Conference Board of the Mathematical Sciences  
*REGIONAL CONFERENCE SERIES IN MATHEMATICS*

supported by the  
National Science Foundation

Number 76

**ALGEBRAIC IDEAS  
IN ERGODIC THEORY**

**Klaus Schmidt**

Published for the  
Conference Board of the Mathematical Sciences  
by the  
American Mathematical Society  
Providence, Rhode Island

Expository Lectures  
from the CBMS Regional Conference  
held at the University of Washington, Seattle  
July 1989

Research supported by National Science Foundation Grants DMS-8814159 and DMS-8820716, as well as additional grants from the IBM Watson Research Center, the IBM Almaden Research Center, and the University of Washington.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 28D05, 28D15, 28D20, 54C70.

---

**Library of Congress Cataloging-in-Publication Data**

Schmidt, Klaus, 1943-

Algebraic ideas in ergodic theory/Klaus Schmidt.

p. cm.—(Regional conference series in mathematics, ISSN 0160-7642; v. 76)

“A series of ten lectures at the Conference Board of the Mathematical Sciences Regional Conference, Seattle, Washington, July 1989.”

Includes bibliographical references and index.

ISBN 0-8218-0727-7

1. Ergodic theory—Congresses. 2. Operator algebras—Congresses.  
3. Markov processes—Congresses. I. Conference Board of the Mathematical Sciences. II. Title.

QA611.7.S36 1990

90-877

515.42—dc20

CIP

---

**Copying and reprinting.** Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy an article for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication (including abstracts) is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Executive Director, American Mathematical Society, P.O. Box 6248, Providence, Rhode Island 02940-6248.

The owner consents to copying beyond that permitted by Sections 107 or 108 of the U.S. Copyright Law, provided that a fee of \$1.00 plus \$.25 per page for each copy be paid directly to the Copyright Clearance Center, Inc., 27 Congress Street, Salem, Massachusetts 01970. When paying this fee please use the code 0160-7642/90 to refer to this publication. This consent does not extend to other kinds of copying, such as copying for general distribution, for advertising or promotional purposes, for creating new collective works, or for resale.

Copyright ©1990 by the American Mathematical Society. All rights reserved.

Printed in the United States of America

The American Mathematical Society retains all rights  
except those granted to the United States Government.

The paper used in this book is acid-free and falls within the guidelines  
established to ensure permanence and durability. ☺

This publication was typeset using  $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$ ,  
the American Mathematical Society's  $\mathcal{T}\mathcal{E}\mathcal{X}$  macro system.

10 9 8 7 6 5 4 3 2 1 95 94 93 92 91 90

## Contents

Introduction .....	1
1. Operator Algebras and Dynamical Systems .....	3
2. Cohomology of Equivalence Relations .....	18
3. Rokhlin's Lemma and Asymptotic Invariance .....	36
4. Dimension .....	43
5. Markov Shifts in Higher Dimensions .....	55
6. Markov Shifts and Markov Groups .....	64
7. The Dynamics of Abelian Markov Groups .....	67
References .....	79
Notation .....	87
Index .....	91

*This page intentionally left blank*

## References

- [AarN] J. Aaronson and M. Nadkarni,  $L_\infty$  eigenvalues and  $L_2$  spectra of nonsingular transformations, *J. London Math. Soc.* **55** (1987), 538–570.
- [Abel] N. H. Abel, *Untersuchung der Functionen zweier unabhängig veränderlicher Größen  $x$  und  $y$ , wie  $f(z, y)$ , welche die Eigenschaft haben, daß  $f(z, f(x, y))$  eine symmetrische Function von  $z, x$  und  $y$  ist*, *J Reine Angew. Math.* **1** (1826), 11–15.
- [AdlM] R. L. Adler and B. Marcus, *Topological entropy and equivalence of dynamical systems*, *Mem. Amer. Math. Soc.* **20** (1979), no. 219.
- [AdlW] R. L. Adler and B. Weiss, *Similarity of automorphisms of the torus*, *Mem. Amer. Math. Soc.*, **98** 1970.
- [Amb] W. Ambrose, *Representations of ergodic flows*, *Ann. of Math.* **42** (1941), 723–739.
- [AmbK] W. Ambrose and S. Kakutani, *Structure and continuity of measurable flows*, *Duke Math. J.* **9** (1942), 25–42.
- [AraW] H. Araki and J. Woods, *A classification of factors*, *Publ. RIMS, Kyoto Univ.*, Ser. A, **4** (1968), 51–130.
- [Aro] D. Z. Arov, *The computation of the entropy for one class of group endomorphisms*, *Zap. Mekh.-Matem. Fakulteta Kharkov Matem.* **30** (1964), 48–69.
- [Ash] J. Ashley, *Bounded-to-1 factors of an aperiodic shift of finite type are 1-to-1 almost everywhere factors also*, preprint, 1989.
- [Atk] G. Atkinson, *Recurrence of co-cycles and random walks*, *J. London Math. Soc.* (2) **13**, 486–488.
- [Bax] R. J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, New York and London, 1982.
- [Bel] R. M. Belinskaya, *Partitioning of a Lebesgue space into trajectories which are definable by ergodic automorphisms*, *Funct. Anal. Appl.* **2** (1968), 4–16.
- [Ber] R. Berger, *The undecidability of the domino problem*, *Mem. Amer. Math. Soc.* No. 66 (1966).
- [Bow1] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, *Lecture Notes in Math.*, Vol. 470, Springer-Verlag, Berlin and New York, 1975.
- [Bow2] R. Bowen, *Anosov foliations are hyperfinite*, *Ann. of Math.* **106** (1977), 549–565.
- [BowF] R. Bowen and J. Franks, *Homology for zero-dimensional non-wandering sets*, *Ann. of Math.* **106** (1977), 73–92.
- [Boyd] D. W. Boyd, *Speculations concerning the range of Mahler's measure*, *Canad. Math. Bull.* **24** (1981), 435–469.
- [BoylMT] M. Boyle, B. Marcus, and P. Trow, *Resolving maps and the dimension group for shifts of finite type*, *Mem. Amer. Math. Soc.* No. 377 (1987).
- [Bra] O. Bratteli, *Inductive limits of finite dimensional  $C^*$ -algebras*, *Trans. Amer. Math. Soc.* **171** (1972), 195–234.
- [ButS] R. Butler and K. Schmidt, *An information cocycle for groups of nonsingular transformations*, *Z. Wahrscheinlichkeitstheorie Verw. Geb.* **69** (1985), 347–360.



- [Con1] A. Connes, *Une classification des facteurs de Type III*, Ann. Sci. École Norm. Sup. **6** (1973), 133–252.
- [Con2] A. Connes, *Classification of injective factors*, Ann. of Math. **104** (1976), 73–115.
- [Con3] A. Connes, *Outer conjugacy classes of automorphisms of factors*, Ann. Sci. École Norm. Sup. (4) **8** (1975), 383–420.
- [Con4] A. Connes, *Periodic automorphisms of the hyperfinite factor of Type  $II_1$* , Acta Sci. Math. **39** (1977), 39–66.
- [Con5] A. Connes, *Sur la théorie non commutative de l'intégration*, Lecture Notes in Math. Vol. 725, pp. 19–143, Springer-Verlag, Berlin and New York, 1979.
- [ConFW] A. Connes, J. Feldman, and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Ergodic Theory Dynamical Systems **1** (1981), 431–450.
- [ConK] A. Connes and W. Krieger, *Measure space automorphisms, the normalizers of their full groups, and approximate finiteness*, J. Funct. Anal. **24** (1977), 336–352.
- [ConT] A. Connes and M. Takesaki, *The flow of weights on factors of Type III*, Tôhoku Math. J. **29** (1977), 473–575.
- [ConW] A. Connes and B. Weiss, *Property T and asymptotically invariant sequences*, Israel J. Math. **37** (1980), 209–210.
- [CorFS] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic Theory*, Springer-Verlag, Berlin and New York, 1982.
- [CovP] E. M. Coven and M. E. Paul, *Endomorphisms of irreducible subshifts of finite type*, Math. Systems Theory **8** (1974), 167–175.
- [CunK] J. Cuntz and W. Krieger, *A class of  $C^*$ -algebras and topological Markov chains*, Invent. Math. **56** (1980), 251–268.
- [Dek] F. M. Dekking, *On transience and recurrence in generalized random walks*, Report 81–20, Delft University of Technology, Delft, 1981.
- [Dix] J. Dixmier, *Les  $C^*$ -algèbres et leurs représentations*, Gauthier-Villars, 1969.
- [DixL] J. Dixmier and C. Lance, *Deux nouveaux facteurs de Type  $II_1$* , Invent. Math. **7** (1969), 226–234.
- [Dye1] H. A. Dye, *On groups of measure-preserving transformations. I*, Amer. J. Math. **81** (1959), 119–159.
- [Dye2] H. A. Dye, *On groups of measure-preserving transformations. II*, Amer. J. Math. **85** (1963), 551–576.
- [Eff] E. G. Effros, *Dimensions and  $C^*$ -algebras*, CBMS Regional Conf. Ser. in Math., Amer. Math. Soc., Providence, RI, 1980.
- [EigS] S. J. Eigen and C. E. Silva, *A structure theorem for  $N$ -to-1 endomorphisms and existence of non-recurrent measures*, preprint, 1988.
- [Eil] G. Elliott, *On the classification of inductive limits of sequences of semi-simple finite dimensional algebras*, J. Algebra **38** (1976), 29–44.
- [Fel] J. Feldman, *Lectures on orbit equivalence*, preprint.
- [FelHM] J. Feldman, P. Hahn, and C. C. Moore, *Orbit structure and countable sections for actions of continuous groups*, Adv. in Math. **28** (1978), 186–230.
- [FelL] J. Feldman and D. A. Lind, *Hyperfiniteness and the Halmos–Rokhlin theorem for non-singular Abelian actions*, Proc. Amer. Math. Soc. **55** (1976), 339–344.
- [FelM1] J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. I*, Trans. Amer. Math. Soc. **234** (1977), 289–324.
- [FelM2] J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. II*, Trans. Amer. Math. Soc. **234** (1977), 325–359.
- [FelR] J. Feldman and A. Ramsay, *Countable sections for free actions of groups*, Adv. in Math. **55** (1985), 224–227.
- [FelSZ] J. Feldman, C. E. Sutherland, and R. J. Zimmer, *Subrelations of ergodic equivalence relations*, Ergodic Theory Dynamical Systems **9** (1989), 239–269.
- [FriO] N. Friedman and D. S. Ornstein, *On the isomorphism of weak Bernoulli transformations*, Adv. in Math. **5** (1970), 365–394.
- [Fur] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in diophantine approximation*, Math. Systems Theory **1** (1967), 1–49.

- [GefG] S. L. Gefger and V. Ya. Golodets, *Ergodic actions with the identity fundamental group*, Dokl. Akad. Nauk. Ukrain. SSR Ser. A **10** (1986), 8–10.
- [GolS] V. Ya. Golodets and S. D. Sinelshchikov, *Existence and uniqueness of cocycles of an ergodic automorphism with dense ranges in amenable groups*, preprint 19–83, FTINT AN USSR, Kharkov, 1–21.
- [Haa] U. Haagerup, *Connes' bicentralizer problem and uniqueness of the hyperfinite factor of Type III<sub>1</sub>*, Acta Math. **158** (1987), 95–148.
- [HamO] T. Hamachi and M. Osikawa, *Ergodic groups of automorphisms and Krieger's theorem*, Sem. Math. Sci., Keio University, Vol. 3, 1981.
- [HamOO] T. Hamachi, Y. Oka, and M. Osikawa, *A classification of non-singular transformation groups*, Mem. Fac. Sci. Kyushu Univ. Ser. A, **28** (1974), 113–133.
- [Han1] D. Handelman,  *$K_0$  of von Neumann algebras and AF  $C^*$ -algebras*, Quart. J. Math. Oxford **29** (1978), 427–441.
- [Han2] D. Handelman, *Positive matrices and dimension groups affiliated to  $C^*$ -algebras and topological Markov chains*, J. Operator Theory **6** (1981), 55–74.
- [Han3] D. Handelman, *Deciding eventual positivity of polynomials*, Ergodic Theory Dynamical Systems **6** (1986), 57–79.
- [Han4] D. Handelman, *Eventually positive matrices with rational eigenvectors*, Ergodic Theory Dynamical Systems **7** (1987), 193–196.
- [HardW] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford University Press, 1979.
- [HarpV] P. de la Harpe and A. Valette, *La propriété (T) de Kazhdan pour les groupes localement compacts*, preprint, 1988.
- [Hed] G. A. Hedlund, *Endomorphisms and automorphisms of the shift dynamical system*, Math. Systems Theory **3** (1969), 320–375.
- [Hop] E. Hopf, *Theory of measure and invariant integrals*, Trans. Amer. Math. Soc. **39** (1932), 373–393.
- [HosMP] B. Host, J. F. Méla, and F. Parreau (in preparation).
- [Hur] W. Hurewicz, *Ergodic theorem without invariant measure*, Ann. of Math. **45** (1944), 192–206.
- [Jon] V. F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), 1–25.
- [JonS] V. F. R. Jones and K. Schmidt, *Asymptotically invariant sequences and approximate finiteness*, Amer. J. Math. **109** (1987), 91–114.
- [Kal] S. Kalikow,  *$T, T^{-1}$  transformation is not loosely Bernoulli*, Ann. of Math. **115** (1982), 393–409.
- [Kam] J. W. Kammeyer, *A complete classification of two-point extensions of a multidimensional Bernoulli shift*, Doctoral Dissertation, University of Maryland, 1988.
- [Kas] P. W. Kasteleyn, *The statistics of dimers on a lattice. I*, Physica **27** (1961), 1209–1225.
- [Kat] I. Katznelson, *Orbit equivalence*, unpublished lecture notes.
- [KatW] I. Katznelson and B. Weiss, *The classification of nonsingular actions, revisited*, Ergodic Theory Dynamical Systems (to appear).
- [Kaz] D. Kazhdan, *Connection of the dual space of a group with the structure of its closed subgroups*, Functional Anal. Appl. **1** (1967), 63–65.
- [KeaS] M. Keane and M. Smorodinsky, *Finitary isomorphisms of irreducible Markov shifts*, Israel J. Math. **34** (1979), 281–286.
- [KitS1] B. Kitchens and K. Schmidt, *Periodic points, decidability, and Markov subgroups*, Lecture Notes in Math, Vol. 1342, pp. 440–454, Springer-Verlag, Berlin and New York, 1988.
- [KitS2] B. Kitchens and K. Schmidt, *Automorphisms of compact groups*, Ergodic Theory Dynamical Systems **9** (1989), 691–735.
- [KitS3] B. Kitchens and K. Schmidt, *Markov subgroups of  $\mathbb{Z}_2^2$* , preprint, 1990.
- [Kre] U. Krengel, *Darstellungssätze für Strömungen und Halbströmungen. II*, Math. Ann. **182** (1969), 1–39.
- [Kri1] W. Krieger, *On nonsingular transformations of a measure space. I*, Z. Wahrscheinlichkeitstheorie Verw. Geb. **11** (1969), 83–97.
- [Kri2] W. Krieger, *On nonsingular transformations of a measure space. II*, Z. Wahrschein-

- lichkeitstheorie Verw. Geb. **11** (1969), 98–119.
- [Kri3] W. Krieger, *On the Araki–Woods asymptotic ratio set and nonsingular transformations of a measure space*, Lecture Notes in Math., Vol. 160, pp. 158–177, Springer-Verlag, Berlin and New York, 1970.
- [Kri4] W. Krieger, *On ergodic flows and the isomorphism of factors*, Math. Ann. **223** (1976), 19–70.
- [Kri5] W. Krieger, *On the finitary isomorphisms of Markov shifts that have finite expected coding time*, Z. Wahrscheinlichkeitstheorie Verw. Geb. **65** (1983), 323–328.
- [Kri6] W. Krieger, *On dimension functions and topological Markov chains*, Invent. Math. **56** (1980), 239–250.
- [Kri7] W. Krieger, *On a dimension for a class of homeomorphism groups*, Math. Ann. **252** (1980), 87–95.
- [Kur] S. Kuratowski, *Topology*, Vol. I. Academic Press, New York and London, 1966.
- [Lam] P.-F. Lam, *On expansive transformation groups*, Trans. Amer. Math. Soc. **150** (1970), 131–138.
- [LaxP] R. Laxton and W. Parry, *On the periodic points of certain automorphisms and a system of polynomial identities*, J. Algebra **6** (1967), 388–393.
- [Led] F. Ledrappier, *Un champ Markovien peut être d'entropie nulle et mélangeant*, C. R. Acad. Sci. Paris, Sér. A **287** (1978), 561–562.
- [Leh] D. H. Lehmer, *Factorization of certain cyclotomic functions*, Ann. of Math. **34** (1933), 461–479.
- [Lie] E. H. Lieb, *Residual entropy of square ice*, Phys. Rev. **162** (1967), 162–172.
- [Lin] D. A. Lind, *Dynamical properties of quasihyperbolic toral automorphisms*, Ergodic Theory Dynamical Systems **2** (1982), 49–68.
- [LinSW] D. A. Lind, K. Schmidt, and T. Ward, *Mahler measure and entropy for commuting automorphisms of compact groups*, Invent. Math. (to appear).
- [LinW] D. A. Lind and T. Ward, *Automorphisms of solenoids and  $p$ -adic entropy*, Ergodic Theory Dynamical Systems **8** (1988), 411–419.
- [LosR] V. Losert and H. Rindler, *Almost invariant sets*, Bull. London Math. Soc. **41** (1981), 145–148.
- [Mac1] G. W. Mackey, *Ergodic theory, group theory, and differential geometry*, Proc. Nat. Acad. Sci. U. S. A. **50** (1963), 1184–1191.
- [Mac2] G. W. Mackey, *Ergodic theory and virtual groups*, Math. Ann. **166** (1966), 187–207.
- [Mah1] K. Mahler, *Eine arithmetische Eigenschaft der Taylor-koeffizienten rationaler Funktionen*, Proc. Acad. Sci. Amsterdam **38** (1935), 50–60.
- [Mah2] K. Mahler, *On some inequalities for polynomials in several variables*, J. London Math. Soc. **37** (1962), 341–344.
- [MarcT] B. Marcus and S. Tuncel, *The weight-per-symbol polytope and scaffolds of invariants associated with Markov chains*, Ergodic Theory Dynamical Systems (to appear).
- [MarP1] N. G. Markley and M. E. Paul, *Matrix subshifts for  $\mathbb{Z}^\nu$  symbolic dynamics*, Proc. London Math. Soc. **43** (1981), 251–272.
- [MarP2] N. G. Markley and M. E. Paul, *Maximal measures and entropy for  $\mathbb{Z}^\nu$  subshifts of finite type*, preprint, 1979.
- [Mas] D. W. Masser, private communication to D. Berend (1985).
- [McD] D. McDuff, *Central sequences and the hyperfinite factor*, Proc. London Math. Soc. **21** (1970), 443–461.
- [MilT] G. Miles and R. K. Thomas, *The breakdown of automorphisms of compact topological groups*. In: *Studies in Probability and Ergodic Theory, Advances in Mathematics Supplementary Studies*, Vol. 2, pp. 207–218, Academic Press, New York and Washington, 1978.
- [Moo] C. C. Moore, *Ergodic theory and von Neumann algebras*, Proc. Symp. Pure Math. **38**, Amer. Math. Soc., Providence, RI, 1982, 179–226.
- [MooS] C. C. Moore and K. Schmidt, *Coboundaries and homomorphisms for nonsingular actions and a problem of H. Helson*, Proc. London Math. Soc. **40** (1980), 443–475.
- [Moz] S. Mozes, *Tilings, substitution systems and dynamical systems generated by them*, M. Sc.

- Dissertation, Hebrew University of Jerusalem, 1987.
- [MurN1] F. J. Murray and J. von Neumann, *On rings of operators*, Ann. of Math. **37** (1936), 116–229.
- [MurN2] F. J. Murray and J. von Neumann, *Rings of operators*. IV, Ann. of Math. **44** (1943), 716–804.
- [Neu] J. von Neumann, *Einige Sätze über meßbare Abbildungen*, Ann. of Math. **33** (1932), 574–586.
- [Oll] J. Moulin-Ollagnier, *Ergodic theory and statistical mechanics*, Lecture Notes in Math., Vol. 1115, Springer-Verlag, Berlin and New York, 1985.
- [OmW1] D. S. Ornstein and B. Weiss, *Ergodic theory of amenable group actions I: the Rohlin theorem*, Bull. Amer. Math. Soc. **2** (1980), 161–164.
- [OmW2] D. S. Ornstein and B. Weiss, *Any flow is an orbit factor of any flow*, Ergodic Theory Dynamical Systems **4** (1984), 105–116.
- [Par1] W. Parry, *Intrinsic Markov chains*, Trans. Amer. Math. Soc. **112** (1964), 55–65.
- [Par2] W. Parry, *Entropy and generators in ergodic theory*, Benjamin, New York–Amsterdam, 1969.
- [Par3] W. Parry, *A note on cocycles in ergodic theory*, Compositio Math. **28** (1974), 343–350.
- [Par4] W. Parry, *Endomorphisms of a Lebesgue space*. III, Israel J. Math. **21** (1975), 167–172.
- [Par5] W. Parry, *The information cocycle and  $\varepsilon$ -bounded codes*, Israel J. Math. **29** (1978), 205–220.
- [Par6] W. Parry, *Finitary isomorphisms with finite expected code-lengths*, Bull. London Math. Soc. **11** (1979), 170–176.
- [Par7] W. Parry, *Synchronisation of canonical measures for hyperbolic attractors*, Commun. Math. Phys. **106** (1986), 267–275.
- [ParS] W. Parry and K. Schmidt, *Natural coefficients and invariants for Markov shifts*, Invent. Math. **76** (1984), 15–32.
- [ParSu] W. Parry and D. Sullivan, *A topological invariant for flows on one-dimensional spaces*, Topology **14** (1975), 297–299.
- [ParT1] W. Parry and S. Tuncel, *Classification problems in ergodic theory*, London Math. Soc. Lecture Note Series, Vol. 67, Cambridge University Press, Cambridge–New York, 1982.
- [ParT2] W. Parry and S. Tuncel, *On the stochastic and topological structure of Markov chains*, Bull. London Math. Soc. **14** (1982), 16–27.
- [PartS] K. R. Parthasarathy and K. Schmidt, *On the cohomology of a hyperfinite action*, Mh. Math. **84** (1977), 37–48.
- [Pim] M. V. Pimsner, *Embedding some transformation group  $C^*$ -algebras into AF-algebras*, Ergodic Theory Dynamical Systems **3** (1983), 613–626.
- [PimV1] M. V. Pimsner and D. V. Voiculescu, *Exact sequences for  $K$ -groups and Ext-groups of certain crossed-product  $C^*$ -algebras*, J. Operator Theory **4** (1980), 93–118.
- [PimV2] M. V. Pimsner and D. V. Voiculescu, *Imbedding the irrational rotation  $C^*$ -algebra into an AF-algebra*, J. Operator Theory **4** (1980), 201–210.
- [Ram] A. Ramsay, *Virtual groups and group actions*, Adv. in Math. **6** (1971), 253–322.
- [Ren] J. Renault, *A groupoid approach to  $C^*$ -algebras*, Lecture Notes in Math., Vol. 793, Springer-Verlag, Berlin and New York, 1980.
- [Rie] M. Rieffel,  *$C^*$ -algebras associated with irrational rotations*, Pacific J. Math. **93** (1981), 415–429.
- [Rob] R. Robinson, *Undecidability and nonperiodicity for tilings of the plane*, Invent. Math. **12** (1971), 177–209.
- [Ros] A. Rosenthal, *Weak Pinsker property and Markov processes*, Ann. Inst. H. Poincaré Probab. Statist. **22** (1986), 347–369.
- [Rud1] D. J. Rudolph, *A two valued step coding for ergodic flows*, Math. Z. **150** (1976), 201–220.
- [Rud2] D. J. Rudolph, *Restricted orbit equivalence*, Mem. Amer. Math. Soc., Vol. 54, 1985.
- [Rud3] D. J. Rudolph,  *$\times 2$  and  $\times 3$  invariant measures and entropy*, Ergodic Theory Dynamical Systems (to appear).
- [Rud4] D. J. Rudolph, *Markov tiling of  $\mathbb{R}^n$  and representations of  $\mathbb{R}^n$  actions*, preprint, 1988.
- [Rue] D. Ruelle, *Thermodynamic Formalism*, Addison-Wesley, Reading, MA, 1978.

- [Sch1] K. Schmidt, *Cocycles on ergodic transformation groups*, MacMillan (Company of India, Ltd., Delhi), 1977.
- [Sch2] K. Schmidt, *Spectra of ergodic group actions*, Israel J. Math. **41** (1982), 151–153.
- [Sch3] K. Schmidt, *Asymptotically invariant sequences and an action of  $SL(2, \mathbb{Z})$  on the 2-sphere*, Israel J. Math. **37** (1980), 193–208.
- [Sch4] K. Schmidt, *Amenability, Kazhdan's property T, strong ergodicity and invariant means for ergodic group actions*, Ergodic Theory Dynamical Systems **1** (1981), 223–236.
- [Sch5] K. Schmidt, *On recurrence*, Z. Wahrscheinlichkeitstheorie Verw. Geb. **68** (1984), 75–95.
- [Sch6] K. Schmidt, *Cohomology and the absence of strong ergodicity for ergodic group actions*, Lecture Notes in Math., Vol. 1132, pp. 486–496, Springer-Verlag, Berlin and New York, 1985.
- [Sch7] K. Schmidt, *Hyperbolic structure preserving isomorphisms of Markov shifts*, Israel J. Math. **55** (1986), 213–228.
- [Sch8] K. Schmidt, *Hyperbolic structure preserving isomorphisms of Markov shifts. II*, Israel J. Math. **58** (1987), 225–242.
- [Sch9] K. Schmidt, *Automorphisms of compact Abelian groups and affine varieties*, Proc. London Math. Soc. (to appear).
- [Sch10] K. Schmidt, *Mixing automorphisms of compact groups and a theorem by Kurt Mahler*, Pacific J. Math. **137** (1989), 371–385.
- [Ser] C. Series, *The Rohlin tower theorem and hyperfiniteness of actions of continuous groups*, Israel J. Math. **30** (1978), 99–122.
- [Sin1] Ja. G. Sinai, *On the concept of entropy of a dynamic system*, Dokl. Akad. Nauk SSSR **124** (1959), 768–771.
- [Sin2] Ya. G. Sinai, *Markov partitions and C-diffeomorphisms*, Functional Anal. Appl. **2** (1968), 64–89.
- [Smy] C. J. Smyth, *On measures of polynomials in several variables*, Bull. Austral Math. Soc. **23** (1981), 49–63.
- [Str] S. Strătilă, *Modular theory in operator algebras*, Abacus Press, Tunbridge Wells, 1981.
- [Sut1] C. E. Sutherland, *Notes on orbit equivalence; Krieger's theorem*, Lecture Notes Ser., Vol. 23, Institute of Mathematics, University of Oslo, Norway, 1976.
- [Sut2] C. E. Sutherland, *A Borel parametrization of Polish groups*, Publ. Res. Inst. Math. Sci., Kyoto Univ., Kyoto **21** (1985), 1067–1086.
- [TemF] H. N. V. Temperley and M. E. Fisher, *Dimer problem in statistical mechanics—an exact result*, Philosophical Magazine **6** (1961), 1061–1063.
- [TemL] H. N. V. Temperley and E. H. Lieb, *Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planer lattices: some exact results for the 'percolation' problem*, Proc. Roy. Soc. London Ser. A. **322** (1971), 251–280.
- [Tom] J. Tomiyama, *On some types of maximal Abelian subalgebras*, J. Funct. Anal. **10** (1972), 373–386.
- [Tot] H. Totoki, *On orbit-equivalence of non-singular flows*, Lecture Notes in Math., Vol. 1021, pp. 687–689, Springer-Verlag, Berlin and New York, 1983.
- [Tro] P. Trow, *Degrees of constant-to-one factor maps*, Proc. Amer. Math. Soc. **103** (1988), 184–188.
- [Tun1] S. Tuncel, *Conditional pressure and coding*, Israel J. Math. **39** (1981), 101–112.
- [Tun2] S. Tuncel, *A dimension, dimension modules, and Markov chains*, Proc. London Math. Soc. **46** (1983), 100–116.
- [Var] V. S. Varadarajan, *Geometry of Quantum Theory (2nd edition)*, Springer-Verlag, Berlin and New York, 1985.
- [Ver1] A. M. Vershik, *Non measurable decompositions, orbit theory, algebras of operators*, Dokl. Akad. Nauk SSSR **199** (1971), 1004–1007.
- [Ver2] A. M. Vershik, *The action of  $PSL(2, \mathbb{Z})$  on  $\mathbb{R}^1$  is approximable*, Uspekhi Mat. Nauk **33** (1978), 209–210.
- [Wag] J. B. Wagoner, *Topological Markov chains,  $C^*$ -algebras, and  $K_2^*$* , Adv. in Math. **71** (1988), 133–185.

- [Wan] H. Wang, *Proving theorems by pattern recognition*. II, Bell Systems Tech. J. **40** (1961).
- [War] T. Ward, Ph.D. Thesis, Warwick, 1989.
- [Wei] B. Weiss, *Orbit equivalence of nonsingular actions*, L'Enseignement Mathématique **29** (1981), 77–107.
- [Wil] R. Williams, *Classification of shifts of finite type*, Ann. of Math. **98** (1973), 120–153; Errata: Ann. of Math. **99** (1974), 380–381.
- [Wol] S. Wolfram, *Universality and complexity in cellular automata*, Physica **10D** (1984), 1–35.
- [Yuz] S. A. Yuzvinskii, *Computing the entropy of a group endomorphism*, Sibirsk. Mat. Z. **8** (1967), 230–239; Engl Transl.: Siberian Math. J. **8** (1968), 172–178.
- [Zim1] R. J. Zimmer, *Hyperfinite factors and amenable ergodic actions*, Invent. Math. **41** (1977), 23–31.
- [Zim2] R. J. Zimmer, *Cocycles and the structure of ergodic group actions*, Israel J. Math. **26** (1977), 214–220.
- [Zim3] R. J. Zimmer, *Amenable ergodic group actions and an application to Poisson boundaries of random walks*, J. Funct. Anal. **27** (1978), 350–372.
- [Zim4] R. J. Zimmer, *Strong rigidity for ergodic actions of semisimple Lie groups*, Ann. of Math. **112** (1980), 511–529.
- [Zim5] R. J. Zimmer, *On the cohomology of ergodic actions of semisimple Lie groups and discrete subgroups*, Amer. J. Math. **103** (1981), 937–950.
- [Zim6] R. J. Zimmer, *Ergodic theory and semisimple Lie groups*, Birkhäuser Verlag, Basel–Boston, Mass., 1984.

*This page intentionally left blank*

## Notation

$\alpha^{\mathcal{N}}$	$\mathbb{Z}^d$ -action defined by a module $\mathcal{N}$	68
$\mathcal{A}(\mathbf{R}), \mathcal{A}'(\mathbf{R})$	von Neumann algebras of a nonsingular equivalence relation	11
$B^1(\mathbf{R}, A)$	set of coboundaries of $\mathbf{R}$ with coefficients in a group $A$	18
$\mathbf{B}(H)$	= the set of bounded linear operators on a Hilbert space $H$	10
$\beta_P$	beta-function of a Markov shift	23
$\mathbb{C}$	= the complex numbers	
$c_B$	restriction of a cocycle $c$ to a set $B$	21
$c_P$	generator of $\Gamma_P/\Delta_P$	23
$\mathcal{C}(\mathbf{R})$	a $C^*$ -algebra of an equivalence relation $\mathbf{R}$	14
$\delta_{a,b}$	= the Kronecker delta ( $\delta_{a,b} = 1$ if $a = b$ , and $\delta_{a,b} = 0$ otherwise)	11
$\Delta_P$	a group associated with a nonnegative matrix $P$	22
$(\mathcal{D}(\mathbf{R}), \mathcal{D}(\mathbf{R})^+)$	dimension module	48
$\mathcal{E}(c), \mathcal{E}^-(c)$	essential range of a cocycle $c$	19
$\mathrm{GL}(n, \mathbb{Z})$	= the group of invertible $n \times n$ matrices with integer entries	40, 72
$\Gamma_P$	a group associated with a nonnegative matrix $P$	22
$h(\cdot)$	entropy	7, 56
$H^1(\mathbf{R}, A)$	first cohomology group of $\mathbf{R}$ with coefficients in an abelian group $A$	18 24
$\mathcal{I}_{\mathbf{R}}: S$	index cocycle of a subrelation	
$k_{\pi}$	= algebraic closure of the prime field $F_{\pi} = \mathbb{Z}_{\pi}$ of characteristic $\pi$	45
$K_0(\mathcal{A})$	dimension group of a $C^*$ -algebra	
$L_V$	nonsingular automorphism of $\mathbf{R}$ defined by an automorphism $V \in [\mathbf{R}]$	13
$L_V, L'_V$	operators in $\mathcal{A}(R)$ and $\mathcal{A}'(R)$ associated with an automorphism $V \in [R]$	11
$\lambda(V)$	module of $V \in \mathcal{N}(R)$	32
$\mu_B$	= restriction of a measure $\mu$ to a measurable set $B$ with $\mu(B) > 0$	
$m_P$	measure of maximal entropy on a Markov shift	7, 8
$\mu_P$	Markov measure on a two-sided Markov shift $X_P$	6
$\mu_{\mathbf{R}}, \mu_{\mathbf{R}}^{(L)}, \mu_{\mathbf{R}}^{(R)}$	measures on a nonsingular equivalence relation $\mathbf{R}$	5
$\mathcal{M}(\mathbf{R}), \mathcal{M}'(\mathbf{R})$	algebra of multiplication operators in $\mathcal{A}'(\mathbf{R})$ and $\mathcal{A}(\mathbf{R})$	11
$\mathbb{N}$	= $\{0, 1, 2, \dots\}$	
$\mathbb{N}^{\times}$	= $\{1, 2, 3, \dots\}$	



$\nu_P$	Markov measure on a one-sided Markov shift $Y_P$	8
$\mathcal{N}(\mathbf{R})$	normalizer of an equivalence relation $\mathbf{R}$	32
$P, \mathbf{P}$	nonnegative, irreducible matrix and the associated stochastic matrix	6
$\pi(V)$	outer period of $V \in \mathcal{N}(R)$	32
$\mathbb{Q}$	= the rationals	32
$\mathbb{R}$	= the real numbers	31
$\mathbb{R}_+$	$[0, \infty) \subset \mathbb{R}$	32
$\mathbb{R}^\times$	$= \mathbb{R} \setminus \{0\}$	31
$\mathbb{R}_+^\times$	$= \mathbb{R}_+ \cap \mathbb{R}^\times$	32
$\Re(\alpha)$	= real part of a complex number $\alpha$	
$[\mathbf{R}]$	full group of an equivalence relation	5
$[[\mathbf{R}]]$	ample group	
$\mathbf{R}_B$	equivalence relation induced on $B$	5
$r(c)$	cohomology invariant	19
$\mathbf{R}^{(c)}$	skew product relation defined by a cocycle $c$	18
$\mathbf{R}^{(c, 1)}, \mathbf{R}^{(c, B)}$	subrelation defined by a cocycle	24
$\mathcal{R}_d$	ring of Laurent polynomials	49
$\mathbf{R}^P, \mathbf{R}'^P$	nonsingular equivalence relations on Markov shifts	6
$[\mathbf{R} : \mathbf{S}]$	index of a subrelation	24
$\mathbf{R}^T$	equivalence relation of a nonsingular group action $T$	5
$\mathbf{R}^V$	equivalence relation of a nonsingular automorphism or endomorphism $V$ )	6, 9
$\mathbf{R}_V$	nonsingular automorphism of $\mathbf{R}$ associated with an automorphism $V \in [\mathbf{R}]$	15
$\rho_{\mathbf{R}, \mu}$	Radon–Nikodym derivative of a nonsingular relation $R$	5
$ S $	= cardinality of a set $S$	5
$\mathbb{S}^1$	$= \{z \in \mathbb{C} :  z  = 1\}$	5
$\mathcal{S}_B$	= a $\sigma$ -algebra $\mathcal{S}$ induced on a set $B$	
$\sigma^{(F, P)}$	higher dimensional Markov shift	56
$\mathrm{SL}(n, \mathbb{Z})$	= the group of $n \times n$ matrices with integer entries and determinant 1	
$\mathbf{S}^P, \mathbf{S}'^P$	nonsingular equivalence relations on Markov shifts	6, 7
$\sigma_P$	Markov shift	6, 7
$\mathcal{S}^T$	= $\sigma$ -algebra of $T$ -invariant subsets in $\mathcal{S}$ , where $T$ is a group action	
$S(f)$	the support of a polynomial $f$	69, 77
$\mathbf{S}^V$	equivalence relation of a nonsingular endomorphism $V$	9
$\mathbb{T}$	$= \mathbb{R}/\mathbb{Z}$	
$T^{(c)}$	group action on a skew product	18
$T^f$	skew product action	28
$\mathcal{T}^{(c)}$	$\mathcal{T}$ -set of a cocycle	20
$W^s(x), W^u(x)$	stable and unstable sets of a point $x$ in a two-sided Markov shift	7
$X_{(F, P)}$	higher dimensional Markov shift space	57
$(X^{\mathcal{N}}, \alpha^{\mathcal{N}})$	dynamical system associated with a module	67
$X_P$	two-sided Markov shift space associated with a nonnegative matrix $P$	6
$X_{P^*}$	set of doubly transitive points in $X_P$	26

$Y_P$	one-sided Markov shift space associated with a nonnegative matrix $P$	8
$\mathbb{Z}$	= the integers	
$\mathbb{Z}/n$	= $\mathbb{Z}/n\mathbb{Z}$	
$Z^1(\mathbf{R}, A)$	set of (1-)cocycles of $\mathbf{R}$ with coefficients in a group $A$	18

*This page intentionally left blank*

## Index

- Adjoint operator, 10
- AF-algebra, 13
- Algebra
  - AF, 14
  - $C^*$ , 10
  - maximal abelian, 10
  - von Neumann, 10
- Allowed
  - map, 56
  - word, 56
- Alphabet, 55
- Amenable
  - equivalence relation, 15
  - group, 15
  - group action, 15
- Ample group, 13
- Araki, 22
- Asymptotically invariant sequence, 38
  - trivial, 38
- Automorphism
  - of a measure space
    - conservative, 6
    - induced, 5
    - measure preserving, 5
    - nonsingular, 4
  - of an equivalence relation, 4, 31
    - inner, 4
- Belinskaya, 31
- Bernoulli action, 36
- Beta function (of a Markov shift), 23
- Borel
  - equivalence relation, 4
  - set, 3
  - space, 3
- $C^*$ -algebra, 10
  - of an equivalence relation, 12
  - of a Markov shift, 12
- Cellular automaton, 61
- Central
  - limit theorem, 21
  - sequence, 38
- Chain recurrence, 46
- Coboundary
  - of an equivalence relation, 18
  - of a group action, 28
- Cocycle, 18
  - bounded, 29
  - cohomologous, 18
  - index, 24
  - induced, 21
  - information, 41
    - of a group action, 28
    - of a Markov shift, 41
  - recurrent, 20
  - transient, 20
- Cohomologous, 18, 28
- Cohomology, 18
  - group, 18
  - invariant, 19
  - lemma, 23
- Commutant (of a self-adjoint set of operators), 10
- Completely positive entropy, 62
- Conditional expectation, 17
- Conjugacy
  - of automorphisms of an equivalence relation, 31
    - outer, 32
  - of group actions, 5
  - of Markov shifts, 68
    - algebraic, 68
    - metric, 7
    - topological, 68
- Connes, 14, 16, 23, 32, 35, 40
- Containment of the trivial representation, 39
- Continued fraction
  - expansion, 10
  - transformation, 9

- Convex hull, 78
- Crossed product, 46
- Crossed product construction, 46
- Cuntz, 14, 55
- Cyclic vector, 11
- Cyclotomic polynomial, 74
  
- Degree (of a factor map), 25
- Descending chain condition, 64
- Dimension
  - of a  $C^*$ -algebra, 45
  - of a factor, 43
  - group
    - of a  $C^*$ -algebra, 45
    - of a Markov shift, 47
  - module, 48
- Dirichlet measure, 29
- Domain (of a semifinite trace), 11
- Doubly transitive point (in a Markov shift), 26
- Dye, 3, 33, 34
  
- Eigenvalue (of a nonsingular automorphism), 28
- Endomorphism
  - countable-to-one, 9
  - measure preserving, 5
  - nonsingular, 4
- Entropy
  - of a Markov shift, 7
  - topological (for  $\mathbb{Z}^d$ -actions), 56
- Equivalence
  - class, 4
  - relation, 4
    - amenable, 15
    - Borel, 4
    - conservative, 20
    - countable (or discrete), 4
    - of an endomorphism, 9
    - of a group action, 4
    - equal (mod  $\mu$ ), 4
    - ergodic, 4
    - finite, 4
    - hyperfinite, 15
    - induced, 5
    - intransitive, 4
    - isomorphic, 4
    - Markov, 25
    - on a Markov shift, 6
    - measure preserving, 5
    - nonsingular, 4
    - on a one-sided Markov shift, 8
    - properly ergodic, 4
    - shift invariant, 6
    - strongly ergodic, 38
    - topological, 12
    - transitive, 4
- Equivalent projections, 43
- Ergodic theorem, 21
- Essential range (of a cocycle), 20
- Expansive
  - group action, 63
  - neighbourhood, 63
- Extension (of an endomorphism to an automorphism), 10
- Extension problem, 56
  
- Factor, 10
  - hyperfinite, 17
  - injective, 16
  - map (of a Markov shift), 25
- Faithful state, 11
- Feldman J., 11, 24
- Flip (on an equivalence relation), Flow
  - equivalence, 53
  - under a function, 34, 60
  - of weights, 35
- Full
  - group, 5
  - shift, 57
- Furstenberg, 61
  
- Golden mean, 58
- Group
  - action, 4
    - amenable, 15
    - Bernoulli, 36
    - conjugate, 5
    - ergodic, 4
    - expansive, 63
    - free, 12
    - measure preserving, 5
    - mixing, 71
    - mixing of order  $r$ , 74
    - nonsingular, 4
    - orbit equivalent, 31
    - properly ergodic, 4
    - transitive, 4
    - weakly mixing, 40
  - amenable, 15
  - ample, 13
  - full, 5
  - Markov, 65
  - measure space construction, 12
  
- Haagerup, 16
- Haar measure, 68
- Homomorphism
  - of  $C^*$ -algebras, 11

- of equivalence relations, 41
- Hopf, 3, 21, 45
- Hurewicz, 21
- Hyperfinite
  - equivalence relation, 15
  - factor, 17
- Index
  - cocycle, 24
  - of a subfactor, 24
  - of a subrelation, 24
- Information cocycle
  - of an equivalence relation, 41
  - of a Markov shift, 41
- Injective factor, 16
- Inner automorphism (of an equivalence relation), 4
- Isomorphism,
  - of  $C^*$ -algebras, 11
  - of equivalence relations, 4
  - of Markov shifts, 7
    - finitary, 7
    - with finite expected code lengths, 7
    - hyperbolic, 7
- Jones, 24
- Krieger, 7, 16, 17, 22, 32, 33, 35, 53, 55
- Laurent polynomial, 49
- Lehmer's problem, 74
- Mackey's program, 3, 21
- Mackey range (of a cocycle), 19
- Mahler measure, 74
- Markov
  - equivalence relation, 25
  - group, 65
  - measure, 6
  - shift, 6, 8
    - higher dimensional, 55
  - shift space
    - higher dimensional, 55
    - one-sided, 8
    - two-sided, 6
- Matrix
  - aperiodic, 6
  - compatible, 7
  - irreducible, 6
  - nonnegative, 6
  - stochastic, 6
- Maximal abelian subalgebra, 10
- Mean (left or right invariant), 15
  - on an equivalence relation, 15
  - on a group, 15
- Measure
  - equivalent, 4
  - ergodic, 4
  - Haar, 68
  - invariant, 5
  - Mahler, 74
  - Markov, 6
  - of maximal entropy, 6, 8
  - quasi-invariant, 5
- Measure space, 3
- Mixing
  - of order  $r$ , 74
  - shape, 75
  - strong, 71
  - weak, 40
- Module, 32, 67
- Moore, C. C., 1
- Multiplication operator, 10
- Murray, 3, 12, 43, 46
- Normal subrelation, 24
- Normalizer
  - of an equivalence relation, 32
  - of  $\mathcal{M}(R)$ , 11
- Operator
  - adjoint, 10
  - multiplication, 10
  - partial isometry, 43
  - projection, 43
  - unitary,
- Orbit equivalence, 31
- Outer
  - aperiodic, 32
  - conjugate, 32
  - period, 32
- Parry, 7, 23
- Periodic point, 56
- Permissible word, 56
- Pimsner, 46
- Poincaré flow
  - of a cocycle, 19
  - of an equivalence relation, 32
- Polynomial
  - (generalized) cyclotomic, 74
  - Laurent, 49
- Prime
  - ideal, 68
    - associated, 68
  - filtration, 69
- Projection, 43
  - equivalent, 43
  - finite, 43
  - infinite, 43

- Property ( $T$ ),
  - for equivalence relations, 39
  - for groups, 39
- Radon-Nikodym derivative, 5
- Recurrence set (of a cocycle), 30
- Reduced primary decomposition, 68
- Representation
  - of an equivalence relation, 39
  - trivial, 39
- Rieffel, 46
- Rigidity theorem, 31
- Rokhlin, 36
  - lemma, 36
  - set, 37
  - tower, 36
- Rotation algebra, 46
- Rudolph, 59, 60
- Saturation, 4
- Self-adjoint subset (of an operator algebra), 10
- Shape, 75
  - mixing, 75
  - nonmixing, 75
  - minimal, 75
  - regularly, 75
- Shift
  - action, 56
  - equivalence, 48
  - invariant, 55
- Skew product, 28
- Stable set, 7
- Standard
  - Borel space, 3
  - measure space, 3
- Strong
  - ergodicity, 38
  - mixing, 71
  - topology (on an operator algebra), 10
- Subgroup
  - cofinite, 71
  - lattice, 31
- Subrelation, 4
  - normal, 24
- Subshift (of finite type), 55
- Support (of a polynomial), 69, 77
- Sutherland, 2
- $T$ -set (of a cocycle), 20
- Takesaki, 35
- Tiling, 59
- Toral automorphism, 64, 72
- Trace, 11
  - normalized, 11
  - semifinite, 11
- Trivial representation, 39
- Tuncel, 23, 48
- Type
  - of an equivalence relation, 22
  - of a factor, 43
- Undecidability, 56
- Unitary
  - operator, 7
  - representation (of an equivalence relation), 39
- Unstable set, 7
- Velocity change, 31
- von Neumann, 3, 11, 14
- von Neumann algebra, 10
  - equivalence relation, 11
  - isomorphic, 11
  - of a free group action, 11
  - of a transitive equivalence relation, 12
- Weak
  - containment of trivial representation, 39
  - mixing, 40
  - Pinsker property, 62
- Weiss, B., 14
- Woods, 22
- Zimmer, 14, 24, 31

ISBN 978-0-8218-0727-9



9 780821 807279

CBMS/76

ISBN 0-8218-0727-7