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# Subfactors and Knots

Vaughan F. R. Jones



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## Introduction

These are the notes of a CBMS series of lectures I gave at Annapolis in the spring of 1988. The lectures were addressed to an audience consisting of low-dimensional topologists and operator algebraists. I tried to make the material comprehensible for both groups. This means that there is an extensive introduction to the theory of von Neumann algebras, and another to knot theory and the braid groups. The material presented in these notes is more or less exactly what was covered in the lectures. One exception is the definition of the knot polynomial  $V(t)$ . In the lectures I began with Kauffman's bracket as a definition and in the notes I end with it. Thus the notes are ordered historically in this respect.

It was a pleasure to give lectures where both knot theory and von Neumann algebras were treated, as well as some elementary material from statistical mechanics and conformal field theory. Since the spring of 1988 the whole area has undergone tremendous development, most notably in terms of the deepening connections with physics. Witten's topological quantum field theory and his invariants for three manifolds have been the most visible part of this work. It was tempting to rewrite the notes to incorporate some of the new developments, but I decided to leave them exactly as they were after the lectures, only adding occasional footnotes with indications of subsequent progress.

Thus some parts of the text seem a little naive, for instance the veiled implication that index for subfactors and central charge of Virasoro representations are directly related. Much progress on these connections has been made by Wassermann.

The choice of topics was, of course, highly personal. Thus the reader will not find much on the detailed classification of subfactors. This is also because the situation was still somewhat unclear in 1988, there being no available proofs of the main results.

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Vaughan Jones  
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