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Sorin Popa

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http://dx.doi.org/10.1090/cbms/086

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Published for the
Conference Board of the Mathematical Sciences
by the
American Mathematical Society
Providence, Rhode Island
with support from the
National Science Foundation

Expository Lectures
from the NSF-CBMS Regional Conference
held at the University of Oregon, Eugene, Oregon
August 24–28, 1993

Research partially supported by
National Science Foundation Grant DMS 8908281

1991 *Mathematics Subject Classification.* Primary 46L35;
Secondary 81E05.

Library of Congress Cataloging-in-Publication Data

Sorin, Popa, 1953–
Classification of subfactors and their endomorphisms / Sorin Popa.
p. cm. — (Regional conference series in mathematics, ISSN 0160-7642; no. 86)
Includes bibliographical references (p. –).
ISBN 0-8218-0321-2
1. Von Neumann algebras. 2. Endomorphisms (Group theory) I. Title. II. Series.
QA1.R33 no. 86
[QA326]
510 s—dc20
[512′.55] 95-17877
CIP

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10 9 8 7 6 5 4 3 2 1 00 99 98 97 96 95

Contents

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Introduction

The purpose of these lecture notes is to provide a more unified and to a certain extent selfcontained presentation of the classification results for inclusions of von Neumann factors of finite Jones index in ([Po2–5]). The classification is in terms of the standard invariant $\mathcal{G}_{\mathcal{N},\mathcal{M}}$ of the subfactor $\mathcal{N} \subset \mathcal{M}$. This invariant is a lattice of inclusions of finite dimensional algebras associated with the Jones iterated basic construction for $\mathcal{N} \subset \mathcal{M}$. It can be recovered from combinatorial data in a similar way inclusions of semisimple algebras can be recaptured from their inclusion diagrams, with the graph of the subfactor, $\Gamma_{\mathcal{N},\mathcal{M}}$, providing most of the information.

We will prove that if $\mathcal{N} \subset \mathcal{M}$ satisfies certain growth conditions, then $\mathcal{G}_{\mathcal{N},\mathcal{M}}$ is a complete invariant for $\mathcal{N} \subset \mathcal{M}$ (up to the isomorphism class of $\mathcal{M}$) and that in fact $\mathcal{N} \subset \mathcal{M}$ is isomorphic to $(\mathcal{N}^{\mathrm{st}} \subset \mathcal{M}^{\mathrm{st}}) \otimes \mathcal{M}$, where $\mathcal{N}^{\mathrm{st}} \subset \mathcal{M}^{\mathrm{st}}$ is the model type II_1 inclusion constructed from $\mathcal{G}_{\mathcal{N},\mathcal{M}}$.

The growth conditions that we have to assume are of two types. First, we require $\Gamma_{\mathcal{N},\mathcal{M}}$ to be strongly amenable, i.e., to be ergodic and to satisfy a certain Følner-type condition. And second, we require $\mathcal{N} \subset \mathcal{M}$ to be approximately inner and centrally free. The condition on $\Gamma_{\mathcal{N},\mathcal{M}}$ is automatically satisfied when $\Gamma_{\mathcal{N},\mathcal{M}}$ is finite, i.e., $\mathcal{N} \subset \mathcal{M}$ has finite depth. The approximate innerness and central freeness are satisfied when $\mathcal{N}$, $\mathcal{M}$ are hyperfinite, provided the modular group of $\mathcal{M}$ is "independent" from $\mathcal{G}_{\mathcal{N},\mathcal{M}}$, a situation that is automatically satisfied if, for instance, $\mathcal{N} \subset \mathcal{M}$ is a finite depth inclusion of hyperfinite type III_1 factors.

Thus, the main application of our general result shows that if an inclusion of hyperfinte type III_1 factors has finite depth then it is isomorphic to its model type II_1 inclusion tensored by a hyperfinite type III_1 factor and it is thus completely determined by $\mathcal{G}_{\mathcal{N},\mathcal{M}}$.

This type III_1 case of the general theorem is what motivated our work. It is the analysis of this case that led us to the consideration of approximate innerness and central freeness for inclusions. Subfactors of finite depth of the hyperfinite type III_1 factor $\mathcal{R}$ appear as ranges of certain endomorphisms σ (or correspondences, in the sense of Connes) with finite statistical parameter $(= (\mathrm{Ind}(\sigma(\mathcal{R}) \subset \mathcal{R}))^{1/2})$ in the theory of superselection sectors of Doplicher, Haag and Roberts (as shown by Longo, [L1, 2], see also [FRS], [Fr]), and in the recent work of Jones and Wassermann ([JW], [W]) relating von Neumann algebras with unitary conformal field theory. We will end our notes by explaining how the classification result for subfactors can in fact be used to show that, at least in theory, endomorphisms of finite depth are uniquely determined, up to outer conjugacy, by certain combinatorial data associated to them ([JPo]).

The notes are divided into six chapters and an Appendix. In the first chapter we recollect some basic facts from the theory of subfactors of finite Jones index, proved in the degree of generality that we need. In it, we also discuss the definition and main properties of the ultrapower and central sequence inclusions of algebras associated to $\mathcal{N} \subset \mathcal{M}$. In Chapters 2 and 3 we consider the notions of approximate innerness and central freeness for $\mathcal{N} \subset \mathcal{M}$, prove some properties and give motivating examples. Chapter 4 contains the proof of the central freeness for hyperfinite subfactors of type III_1, with the subfactor versions of Haagerup's bicentralizer theorem and Connes-Takesaki relative commutant theorem as main technical results. Chapter 5 contains the proof of the general classification result (see Theorem 5.1 for the statement). It is achieved through a sequence of steps that one could label as "localization-globalization-exhaustion" techniques. Chapter 6 contains applications of the main classification result to the actual listing of subfactors of index ≤ 4, by using the results on the structure and description of the invariants $\mathcal{G}_{\mathcal{N},\mathcal{M}}$ in ([Oc2], [Iz], [IzKa], [SuVa]), and to the classification of automorphisms and endomorphisms. This part can be read independently. The Appendix contains a detailed proof of the "local quantization principle" which plays a crucial role in the proofs of the results in these notes, as it did in fact in ([Po1–4, 7]) as well.

The reader is warned that the terms "unified" and "selfcontained" that we used at the beginning of this presentation are referring to the papers [Po1–5] only (and still, there are a couple of combinatorial results from [Po4] that are stated here without proofs). But in order to read the material in these notes one needs to have a solid background on (or read it in parallel with) a number of papers that are quoted in the text, notably: Jones' index for subfactor papers [J2, 3], Tomita-Takesaki theory, the work of Powers and Araki-Woods on type III factors [P], [AW], Connes' discrete decomposition and classification of injective factors and their automorphisms, [C1–4], the Connes-Størmer transitivity theorem [CS], the Connes-Takesaki continuous decomposition [CT], Takesaki duality [T1], Haagerup's spatial theory, bicentralizer theorem and uniqueness of the hyperfinite type III_1 factor [H1, 2].

Finally, we mention that we do not discuss in these notes the characterization and combinatorial properties of the invariants $\mathcal{G}_{\mathcal{N},\mathcal{M}}$, $\Gamma_{\mathcal{N},\mathcal{M}}$, for which we refer the reader to ([Oc2], [H3], [Po10]).

The presentation of the material follows the CBMS series of lectures that I gave on this subject in Eugene, Oregon, August 1993. I would like to thank C. Phillips and all others involved in the organization of that CBMS series.

It is a pleasure for me to thank A. Connes, V. F. R. Jones, P. Loi, M. Takesaki and A. Wassermann for their comments, suggestions and stimulating discussions, extended to me during the write up of these notes.

My special thanks are due to L. Martin for her excellent typing work on this paper.

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ISBN: 0-8218-0321-2

9 780821 803219