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To my mother and father

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Preface

The last twenty years have seen an active interaction between mathematics and physics. This book is devoted to one of the new areas which deals with mathematical structures related to conformal field theory and its q-deformatons. We discuss the interplay between Knizhnik-Zamolodchikov type equations, the Bethe ansatz method, representation theory, and geometry of multi-dimensional hypergeometric integrals.

This book aims to provide an introduction to the area and expose different facets of the subject. The book contains constructions, discussions of notions, statements of theorems, and illustrating examples. The exposition is restricted to the simplest case of the theory associated with the Lie algebra $\mathfrak{sl}_2$.

The book consists of six chapters.

In Chapter 1 the Knizhnik-Zamolodchikov equation is realized as a Gauss-Manin connection, that is as the differential equation for multi-dimensional hypergeometric integrals.

In Chapter 2 the space of hypergeometric solutions of a Knizhnik-Zamolodchikov equation is identified with the multiplicity space of the tensor product of representations of the corresponding quantum group. This identification allows one to identify the monodromy group of the Knizhnik-Zamolodchikov equation with the R-matrix representation of the braid group.

In Chapter 3 determinant formulas for hypergeometric integrals are discussed. The determinant formulas allow one to conclude that the hypergeometric solutions give all solutions to the Knizhnik-Zamolodchikov equation. The second topic of Chapter 3 is the dynamical equations. The dynamical equations are equations complementary to the Knizhnik-Zamolodchikov equations. In the simplest example the equations respectively become the Gauss contiguous function relations and the Euler hypergeometric differential equation for the classical hypergeometric function.

In Chapter 4 the relations between the Bethe ansatz method and quasiclassical asymptotics of hypergeometric functions are discussed.

Chapter 5 is devoted to elliptic hypergeometric functions, their modular properties, and relations to Macdonald polynomials and traces of intertwining operators.

In Chapter 6 we explain how the geometric theory of the Knizhnik-Zamolodchikov differential equation has to be quantized to give solutions to the quantum Knizhnik-Zamolodchikov difference equation.

The book grew out of graduate courses at UNC in the fall of 2001, at MIT in the spring of 2002, and ten lectures given by the author at the NSF-CBMS Conference "Arrangements and Mathematics Physics", at Louisiana State University, Baton Rouge, during the week of January 11-15, 2002.

I am grateful to students in my courses: A. Boysal, M. Graña, S. Lau, I. Mencattini, A. Oblomkov, V. Ostrik, L. Stevens, J. Scott, and to my colleagues: I. Cherednik, D. Cohen, J. Damon, P. Eberlein, P. Etingof, G. Felder, Y. Markov, E. Mukhin, P. Orlik, I. Scherbak, V. Tarasov, H. Terao for many valuable discussions. My special thanks are to Daniel Cohen who organized the conference in Baton Rouge.

Alexander Varchenko
Chapel Hill, March of 2003

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