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CHAPTER 3

Localization of Newton and Riesz Potentials

3.1. Localization lemmas

Recall that Newton's kernel is defined by $\Phi(x) = -\frac{a_n}{|x|^{n-2}}$, $n \geq 3$, where $a_n > 0$ is a constant, $\Phi(x) = -\frac{1}{2\pi}\log\frac{1}{|x|}$ for $n = 2$. Let $\Psi = \nabla\Phi$; this is a Riesz kernel (actually it is a vector Riesz kernel consisting of n Riesz kernels Ψ^i which are $\partial_i\Phi := \frac{\partial}{\partial x_i}\Phi$). These are Calderón–Zygmund (CZ) kernels having singularity of order $n-1$. Sometimes we use the notation

$$W^S(x) = (\Phi * S)(x),\ V^S(x) = (\Psi * S)(x)\,,$$

for the Newton and Riesz potentials of distribution S, respectively.

We always consider E in $B_0 := B(0,1)$. The letter S is reserved for distributions with compact support, dx denotes Lebesgue measure in $\mathbb{R}^n$. The letter A denotes any absolute constant or a constant that depends only on n; $|\cdot|$ stands for the norm in $\mathbb{C}^n$.

LEMMA 3.1. *Let S be supported on E and $\Phi * S \in L(E,1)$. Let $B = B(x,r)$ be a ball, and let $\varphi \in C_0^\infty(2B)$ be such that $\nabla^2\varphi \leq \frac{A}{r^2}$. Then*

$$|\langle S,\varphi\rangle| \leq A\,r^{n-1}\,. \tag{3.1}$$

Proof. Let $f = \Phi * S \in L(E,1)$. Then $|f(y) - f(x)| \leq 2r\,\forall y \in B(x,2r)$. Now $|\langle S,\varphi\rangle| = |\langle\Delta f,\varphi\rangle| = |\langle\Delta(f - f(x)),\varphi\rangle| = |\langle(f - f(x)),\Delta\varphi\rangle| \leq \frac{A\,r}{r^2}\mathrm{vol}(B(x,r)) \leq A\,r^{n-1}$. □

Consider the following situation. Distribution S is supported on $E \subset B := B(x,r)$, $\Phi * S \in L(E,1)$.

LEMMA 3.2. *Let φ be from $C^\infty(2B)$, and*

$$\|\varphi\|_{L^\infty(2B)} \leq Dr,\ \|\nabla\varphi\|_{L^\infty(2B)} \leq D,\ \|\nabla^2\varphi\|_{L^\infty(2B)} \leq \frac{D}{r}\,.$$

Then

$$|\langle S,\varphi\rangle| \leq ADr^n\,, \tag{3.2}$$

and moreover

$$|\langle S,\varphi\rangle| \leq ADr\,\gamma(E)\,. \tag{3.3}$$

Proof. Consider any $g \in C_0^\infty(2B)$ such that $g = 1$ on B, $0 \leq g \leq 1$, $\nabla^2 g \leq \frac{A}{r^2}$. Let $f = \Phi * S \in L(E,1)$. Then $|f(y) - f(x)| \leq 2r\,\forall y \in B(x,2r)$.

Put $F := \varphi g(f - f(x))$. Then

$$\Delta F = \varphi g S + \Delta(\varphi g)(f - f(x)) + \langle\nabla(\varphi g), V * S\rangle =: I + II + III\,.$$

Notice that II and III are bounded functions supported on $2B$. Hence $F = \Phi * I + \Phi * IIdx + \Phi * IIIdx + H$, where H is harmonic in $\mathbb{R}^n$ and vanishes at infinity. So $H = 0$. Therefore,

$$\Psi * I = \nabla F - \Psi * IIdx - \Psi * IIIdx\,. \tag{3.4}$$

Let us estimate functions II and III. $\|II\|_{L^\infty(2B)} \leq \|\nabla^2\varphi\|\|g\|\|f - f(x)\| + \|\nabla\varphi\|\|\nabla g\|\|f - f(x)\| + \|\varphi\|\|\nabla^2 g\|\|f - f(x)\| \leq A\frac{D}{r}r + A\,D\frac{A}{r}r + A\,Dr\frac{A}{r^2}r \leq A\,D$. Similarly, $\|III\|_{L^\infty(2B)} \leq A\,D$. As II and III are supported on $2B$, we now get

$$\|\Psi * IIdx\|_{L^\infty(2B)} \leq A\,D\,r,\ \|\Psi * IIIdx\|_{L^\infty(2B)} \leq A\,D\,r\,.$$

So by the maximum value theorem for harmonic functions,

$$|\Psi * IIdx|(y) \leq A\,D\,r,\ |\Psi * IIIdx|(y) \leq A\,D\,r\ \forall y \in \mathbb{R}^n\,. \tag{3.5}$$

The first term in (3.4) is zero outside of $2B$. Inside

$$|\nabla F(y)| \leq |g\nabla\varphi(f - f(x)| + |\varphi\nabla g(f - f(x)| + |\varphi g\,\Psi * S| \leq A\,D\,r. \tag{3.6}$$

Gathering (3.4), (3.5), (3.6), we get

$$|(\Psi * \varphi S)(y)| = |(\Psi * \varphi g S)(y)| \leq A\,D\,r\ \forall y \in \mathbb{R}^n\,. \tag{3.7}$$

Consider now a new distribution $S_\varphi := \varphi S/ADr$. Then (3.7) shows that

$$S_\varphi \in L(E,1)\,.$$

Apply Lemma 3.1 to S_φ. Then

$$|\langle S_\varphi, g\rangle| \leq A\,r^{n-1}\,.$$

This implies

$$|\langle S, \varphi\rangle| = |\langle S, \varphi g\rangle| = ADr|\langle S_\varphi, g\rangle| \leq A\,D\,r^n\,.$$

This is (3.2). But actually, the definition of $\gamma(E)$ and (3.7) shows that

$$|\langle S_\varphi, 1\rangle| \leq \gamma(E)\,.$$

But this is the same as

$$|\langle S, \varphi\rangle| = A\,D\,r\,|\langle S_\varphi, 1\rangle| \leq A\,D\,r\,\gamma(E)\,,$$

which is (3.3).

□

Recall that we consider the following situation. Distribution S is supported on $E \subset B := B(x,r)$, $\Phi * S \in L(E,1)$. Consider Lemma 3.2 with a special φ. Fix a $z \in \mathbb{R}^n \setminus 3B$ and put

$$\varphi_z(y) := \Phi(z - y) - \Phi(z - x)\,.$$

Notice that it satisfies the assumptions of Lemma 3.2 with $D = \frac{1}{\mathrm{dist}(z,E)^{n-1}}$. Thus, we get the following corollary.

COROLLARY 3.3. *If distribution S is supported on $E \subset B := B(x,r)$ and $\Phi * S \in L(E,1)$, then $|(\Phi * S)(z) - \langle S, 1\rangle\Phi(z - x)| = |\langle S, \varphi_z(y)\rangle|$ and therefore*

$$|(\Phi * S)(z) - \langle S, 1\rangle\Phi(z - x)| \leq \frac{A\,r\,\gamma(E)}{\mathrm{dist}(z,E)^{n-1}}\,. \tag{3.8}$$

In particular, if $\langle S, 1\rangle = 0$, *we get*

$$|(\Phi * S)(z)| \leq \frac{A\, r\, \gamma(E)}{\operatorname{dist}(z, E)^{n-1}}\,, \tag{3.9}$$

and

$$|(\Psi * S)(z)| \leq \frac{A\, r\, \gamma(E)}{\operatorname{dist}(z, E)^{n}}\,. \tag{3.10}$$

3.2. A building block for the construction of special measures

Consider again the following situation. Distribution S is supported on $E \subset B := B(x, r)$, $\Phi * S \in L(E, 1)$. Let us modify distribution S. First of all, consider measure μ, which is a surface measure on the sphere of radius $R = c_n \gamma(E)^{\frac{1}{n-1}}$ centered at x. The constant c_n is chosen in such a way that $R \leq r/n$ and $\|\mu\| \asymp \gamma(E)$, where the constants of comparison depend only on dimension n. Let

$$\hat{S} = \langle S, 1\rangle \frac{\mu}{\|\mu\|}\,.$$

Of course, complex measure $\hat{S}$ satisfies

$$\langle \hat{S}, 1\rangle = \langle S, 1\rangle\,. \tag{3.11}$$

Also, (3.8) can be rewritten as

$$|(\Phi * S)(z) - (\Phi * \hat{S})(z)| \leq \frac{A\, r\, \gamma(E)}{\operatorname{dist}(z, E)^{n-1}}\ \forall z \in \mathbb{R}^n \setminus 3B\,. \tag{3.12}$$

3.3. Localization on special cubes

Let $E \subset B_0 := B(0, 1)$. We consider the family of cubes with the following properties:

$$\gamma_+(\cup_{i=1}^N Q_i) \leq C_0 \gamma_+(E)\,, \tag{3.13}$$

$$\sum_{i=1}^N \gamma_+(2Q_i \cap E) \leq C_1 \gamma_+(E)\,, \tag{3.14}$$

$$\operatorname{diam} Q_i \leq \frac{1}{10} \operatorname{diam} E\,. \tag{3.15}$$

Cubes $\{Q_i\}$ will have bounded multiplicity of overlapping. Constants C_0, C_1 depend only on the dimension n. Such a family of cubes will be constructed below. Unlike other constants depending only on the dimension, we keep special names for these constants because of their special part in the proof.

Consider $\{g_i\}_{i=1}^N$, $g_i \in C_0^\infty(2Q_i)$, $0 \leq g_i \leq 1$ on $\mathbb{R}^n$, $|\nabla^2 g_i| \leq \frac{A}{\ell(Q_i)^2}$. In addition,

$$\sum_{i=1}^N g_i = 1 \ \text{ on } \ \cup_{i=1}^N Q_i\,. \tag{3.16}$$

Let $\Phi * S \in L(E, 1)$, $E \subset B_0 := B(0, 1)$.

LEMMA 3.4. *There exists* $A < \infty$ *(depending only on the dimension) such that* $\frac{1}{A}\Phi * g_i S \in L(E, 1)$. *Henceforth,* $\frac{1}{A}\Phi * g_i S \in L((2Q_i \cap E), 1)$.

Proof. Fix i. Let x be the center of Q_i. Put $f := \Phi * S$ and $F := g_i(f - f(x))$. Then $\Delta F = g_i S + \Delta g_i(f - f(x)) + \langle \nabla g_i, \Psi * S\rangle =: I + II + III$. Hence

$$F = \Phi * g_i S + \Phi * II dx + \Phi * III dx + H\,,$$

where H is harmonic in $\mathbb{R}^n$. Notice that II, III are supported on $2Q_i$, bounded and $\|II\|_{L^\infty} \leq \frac{A}{\ell(Q_i)}, \|III\|_{L^\infty} \leq \frac{A}{\ell(Q_i)}$. Therefore, the first three terms on the right hand side of the previous equality vanish at infinity. Its left hand side F vanishes outside $2Q_i$, so H vanishes at infinity, and this means that $H = 0$. Now we can write

$$\Psi * g_i S = \nabla F - \Psi * II dx - \Psi * III dx\,. \tag{3.17}$$

We already saw that II, III are such that $\|II\|_{L^\infty} \leq \frac{A}{\ell(Q_i)}, \|III\|_{L^\infty} \leq \frac{A}{\ell(Q_i)}$, and they are supported by $2Q_i$. In particular, $|(\Psi * II dx)(y)| \leq A$, $|(\Psi * III dx)(y)| \leq A$, $\forall y \in 2Q_i$, and then, by the maximal principle,

$$|(\Psi * II dx)(y)| \leq A,\ |(\Psi * III dx)(y)| \leq A,\ \forall y \in \mathbb{R}^n\,. \tag{3.18}$$

The first term ∇F in (5.62) is also uniformly bounded. In fact, $\nabla F = \nabla g_i(f - f(x)) + g_i \Psi * S$. The second term here is bounded by the assumption: $\Phi * S \in L(E,1)$. The first term is bounded because we have to consider it only on $2Q_i$ (outside it is zero) and on $2Q_i$ $|f - f(x)| \leq A\ell(Q_i)$ (again by the fact that $\Phi * S \in L(E,1)$).

Finally, (3.17), (3.18) and the uniform boundedness of ∇F imply that $\Psi * g_i S$ is bounded in $\mathbb{R}^n$. The lemma is proved. □

Now we obviously have

COROLLARY 3.5. *Let* $E \subset B(0,1)$, $\Phi * S \in L(E,1)$, $\{Q_i\}$ *satisfy* (3.13)-(3.15). *Then*

$$|\langle S, g_i\rangle| \leq A\,\gamma(2Q_i \cap E)\,. \tag{3.19}$$

3.4. Modification of distribution S. Construction of auxiliary measures

Let us recall that we have potentials $W = \Phi *, V = \Psi *$. Let us combine Lemma 3.2 and Lemma 3.4 to build special measures μ, ν starting with distribution S such that

$$\Phi * S \in L(E,1)\,.$$

We fix $i = 1, \ldots, N$ and consider the distribution $g_i S$ which is supported on $2Q_i$. This set plays the role of $2B(x,r)$ from Lemma 3.2, and $g_i S$ plays the role of S from this lemma. We use the construction from Section 3.2 with $g_i S$ playing the role of S. We already proved (see (3.12)) that

$$|W^{g_i S}(z) - W^{\widehat{g_i S}}(z)| \leq \frac{A\,\ell(Q_i)\gamma(2Q_i \cap E)}{\operatorname{dist}(z, Q_i)^{n-1}}\ \ \forall z \in \mathbb{R}^n \setminus 3Q_i\,, \tag{3.20}$$

and consequently

$$|V^{g_i S}(z) - V^{\widehat{g_i S}}(z)| \leq \frac{A\,\ell(Q_i)\gamma(2Q_i \cap E)}{\operatorname{dist}(z, Q_i)^{n}}\ \ \forall z \in \mathbb{R}^n \setminus 3Q_i\,. \tag{3.21}$$

Also by Lemma 3.4,

$$|V^{g_i S}(z)| \leq A\ \ \forall z \in \mathbb{R}^n\,. \tag{3.22}$$

Recall that we have a sphere $\Sigma_i(x_i, R_i)$ lying inside Q_i centered at the center x_i of Q_i, such that its surface measure $\mathcal{H}^{n-1}(\Sigma_i(x_i, R_i))$ is comparable (with constants depending only on the dimension) to $\gamma(2Q_i \cap E)$, and we introduced measures

$$\mu_i := \mathcal{H}^{n-1}|\Sigma_i(x_i, R_i) \;\; \mu := \sum \mu_i \,.$$

Now let us replace the distribution $S = \sum g_i S$ by complex measure

$$\nu := \sum \widehat{g_i S} = \sum \langle S, g_i \rangle \frac{\mu_i}{\|\mu_i\|} \,. \tag{3.23}$$

Now (3.19) gives that

$$\|\frac{d\nu}{d\mu}\|_\infty \leq \hat{A} \,. \tag{3.24}$$

We know that $|V^S| \leq 1$ in $\mathbb{R}^n$. Our next goal is to see what kind of boundedness we have for the potential of the modified measure ν. We will "almost prove" that

$$\int |V^\nu| d\mu \leq A\|\mu\| \,.$$

3.5. Ahlfors balls

Lemma 3.1 proves (we should choose $r = \mathrm{diam}(E)$ and $\varphi = 1$ on E in Lemma 3.1) the existence of the constant $A(n) < \infty$ such that

$$\gamma(E) \leq A_1(n)(\mathrm{diam}(E))^{n-1} \,. \tag{3.25}$$

Recall that there exists a positive finite constant $A_2(n)$ such that

$$A_2^{-1}\|\mu_i\| \leq \gamma(2Q_i \cap E) \leq A_2\|\mu_i\| \,. \tag{3.26}$$

Let us use these constants $A_1(n), A_2(n)$ along with C_1 from (3.14) to introduce

$$C_2 = 100^{n-1} A_1(n) A_2(n)\, C_1 \,. \tag{3.27}$$

Let us also use from now on the notation

$$F := \cup_{i=1}^N Q_i \,.$$

We have $\mu(F) = \sum \|\mu_i\| \leq A_2 \sum \gamma(2Q_i \cap E)$, and so

$$\mu(F) \leq C_1 A_2 \gamma(E) \leq C_1 A_2 \gamma(F) \leq C_1 A_1 A_2 \,\mathrm{diam}(F)^{n-1} \,.$$

This immediately implies

$$\forall x \in F \; \forall R > \frac{\mathrm{diam}\, F}{100}, \;\; \mu(B(x,R)) \leq C_2 R^{n-1} \,. \tag{3.28}$$

Now we are ready to introduce the notion of a non-Ahlfors ball. The ball $B(x,R), x \in F$ is called non-Ahlfors if $\mu(B(x,r)) > C_2 R^{n-1}$. Otherwise it is called an Ahlfors ball. We see that every ball centered at $x \in F$ is an Ahlfors ball if its radius is larger than $\frac{\mathrm{diam}\, F}{100}$. Denote by $R(x)$ the supremum of all non-Ahlfors balls centered at x. Ahlfors points are those for which $R(x) = 0$. We know that $\forall x \; R(x) \leq \frac{\mathrm{diam}\, F}{100}$.

Denote

$$H_0 := \cup_{x \in F} B(x, R(x)) \,.$$

By Vitali's covering lemma choose disjoint $B(x_k, R(x_k))$ and put

$$H = \cup_k B(x_k, 3R(x_k)) \,.$$

Then all non-Ahlfors balls are contained in H and

$$\operatorname{dist}(x, F \setminus H) \geq R(x), \ \forall x \in H \cap F \,, \tag{3.29}$$

$$\sum_k R(x_k)^{n-1} \leq \frac{1}{C_2}\mu(F) \,. \tag{3.30}$$

3.6. The principal estimate for auxiliary measures

Let us set

$$V_*^\nu(y) := \sup_{\varepsilon>0} |(\Psi * \chi_{\mathbb{R}^n \setminus B(y,\varepsilon)} d\nu)(y)| \,.$$

Following [**64**], we prove

THEOREM 3.6.

$$\int_{F\setminus H} V_*^\nu(y) d\mu(y) \leq C(C_2)\mu(F) \,. \tag{3.31}$$

Proof. Let ψ be a bell-like function with compact support, $\psi_\varepsilon(x) = \frac{1}{\varepsilon^n}\psi(\frac{x}{\varepsilon})$, and

$$U_\varepsilon^\sigma := \psi_\varepsilon * V^\sigma, \ \ U_*^\sigma = \sup_\varepsilon |U_\varepsilon^\sigma| \,.$$

The kernel of U_ε is $k_\varepsilon := \psi_\varepsilon * \frac{x}{|x|^n}$ (here x is a vector $(x_1, \ldots, x_n)$), and so

$$k_\varepsilon(x) = \frac{x}{|x|^n}, \ \text{if } |x| > \varepsilon \text{ and } |k_\varepsilon(x)| \leq \frac{A}{\varepsilon^{n-1}} \text{ otherwise} \,. \tag{3.32}$$

In particular, given a complex measure ν, we introduce the maximal function

$$M\sigma(x) := \sup_r \frac{|\sigma|(B(x,r)}{r^{n-1}}$$

and write

$$|U_\varepsilon^\sigma(x) - V_\varepsilon^\sigma(x)| \leq AM\sigma(x) \,. \tag{3.33}$$

Remark. Maximal function M will be used only for measures (and, of course, never for distributions).

LEMMA 3.7.

$$U_*^{g_i S - \widehat{g_i S}}(z) \leq \frac{A\ell(Q_i)\gamma(2Q_i \cap E)}{\operatorname{dist}(z, 2Q_i)^n} \ \ \forall z \in \mathbb{R}^n \setminus 4Q_i \,. \tag{3.34}$$

Let us postpone the proof of Lemma 3.7 and finish now the proof of Theorem 3.6. We start with

$$\int_{F\setminus H} V_*^\nu d\mu = \int_{F\setminus H} (U_*^\nu - V_*^\nu) d\mu + \int_{F\setminus H} U_*^\nu d\mu =: I + II \,.$$

Let us use the definition of H, (3.27) together with (3.33) and (3.24) to write

$$|I| \leq A \int_{F\setminus H} M\nu d\mu \leq A\hat{A} \int_{F\setminus H} M\mu d\mu \leq A\hat{A}C_2\mu(F) \,. \tag{3.35}$$

To estimate II we write

$$\int_{F\setminus H} U_*^\nu d\mu \leq \int_{F\setminus H} U_*^S d\mu + \int_{F\setminus H} U_*^{S-\nu} d\mu \,. \tag{3.36}$$

Recall that $|V^S(y)| \leq A$ everywhere, hence its convolution with ψ_ε is bounded, and so $U^S_* = \sup_\varepsilon |\psi_\varepsilon * V^S|$ is bounded everywhere by a constant A. So the first integral in (3.36) is bounded by $A\,\mu(F)$. To estimate,

$$\int_{F\setminus H} U_*^{S-\nu} d\mu \leq \sum_{i=1}^N \int_{F\setminus H} U_*^{g_iS-\widehat{g_iS}} d\mu$$

$$\sum_{i=1}^N \Big(\int_{4\,Q_i} \cdots + \int_{F\setminus(4Q_i\cup H)} \cdots \Big) := \sum_{i=1}^N (T_{i1} + T_{i2})\,.$$

Let $\alpha_i := g_iS - \widehat{g_iS}$. We want to estimate $|(\psi_\varepsilon * V^{\alpha_i})(z)|, z \in 4Q_i$. We know that functions V^{g_iS} are uniformly bounded. Just use (3.22). Let us prove now that

$$\|V^{\widehat{g_iS}}\|_{L^\infty(\mathbb{R}^n)} \leq A\,. \tag{3.37}$$

Let Σ be a sphere of radius R centered at x on which measure $\widehat{g_iS}$ is concentrated. Let σ denote the normalized surface measure on Σ. It is easy to see that

$$V^\sigma(z) = 0 \;|z-x| < R,\;\; V^\sigma(z) = \frac{z-x}{|z-x|^n},\;\; |z-x| > R\,.$$

If we use (3.19),(3.26), we get $|\langle S, g_i\rangle| \leq A(\gamma(2Q_i \cap E))^{n-1} \leq A\,R^{n-1}$. But the calculation of V^σ above gives $|V^\sigma| \leq \frac{A}{R^{n-1}}$ almost everywhere. Then using $|V^{\widehat{g_iS}}(z) = |\langle S, g_i\rangle||V^\sigma(z)|$, we get (3.37).

In particular,

$$T_{i1} \leq A\,\mu(4Q_i)\,. \tag{3.38}$$

To estimate T_{i2} we use Lemma 3.7. Let N be the least integer such that $B_N := (4^{N+1}Q_i \setminus 4^N Q_i) \setminus H \neq \emptyset$. Then

$$T_{i2} \leq A\ell(Q_i)\mu(Q_i) \sum_{k=N}^\infty \int_{B_N} \frac{d\mu(z)}{\operatorname{dist}(z, 2Q_i)^n}\,.$$

We continue by

$$T_{i2} \leq A\,\ell(Q_i)\,\mu(Q_i) \sum_{k=N}^\infty \frac{\mu(4^{k+1}Q_i)}{4^{kn}\ell(Q_i)^n}\,.$$

Now let $z_0 \in B_N$. Then for $k \geq N$ we have

$$\mu(4^{k+1}Q_i) \leq \mu(B(z_o, A\ell(4^{k+1}Q_i))) \leq A\,4^{k(n-1)}\ell(Q_i)^{n-1}\,.$$

We used here the fact that $z_0 \notin H$. We can continue the estimate:

$$T_{i2} \leq A\,\ell(Q_i)\,\mu(Q_i)\,\frac{1}{\ell(4^N Q_i)} \leq A\mu(Q_i)\,. \tag{3.39}$$

Combining (3.38), (3.39) one obtains

$$\int_{F\setminus H} U_*^{S-\nu} d\mu \leq A \sum_{i=1}^N \mu(4Q_i)\,. \tag{3.40}$$

Combining with (3.35), (3.36), (3.38) and using the fact of finite overlapping of $4Q_i$, one gets

$$\int_{F\setminus H} V_*^\nu d\mu \leq A\,\mu(F)\,.$$

Theorem 3.6 is completely proved. □

Let us prove now Lemma 3.7.

Proof. Fix $z \in \mathbb{R}^n \setminus 4Q_i$ and let $\varepsilon \leq \frac{\operatorname{dist}(z,2Q_i)}{2}$. Then for any $y \in B(z,\varepsilon)$, we have $\operatorname{dist}(y,2Q_i) \asymp \operatorname{dist}(z,2Q_i)$, and every such y is outside of $3Q_i$. Therefore, for all $y \in B(z,\varepsilon)$ we have (3.21):

$$|V^{g_i S}(y) - V^{\widehat{g_i S}}(y)| \leq \frac{A\,\ell(Q_i)\gamma(2Q_i \cap E)}{\operatorname{dist}(z,Q_i)^n}\,. \tag{3.41}$$

Making the convolution with ψ_ε, we get

$$|U_\varepsilon^{g_i S - \widehat{g_i S}}(z)| \leq \frac{A\,\ell(Q_i)\gamma(2Q_i \cap E)}{\operatorname{dist}(z,Q_i)^n}\,. \tag{3.42}$$

Now fix $z \in \mathbb{R}^n \setminus 4Q_i$ and let $\varepsilon > \frac{\operatorname{dist}(z,2Q_i)}{2} > \ell(Q_i)$. Put $\alpha_i := g_i S - \widehat{g_i S}$. Then

$$U_\varepsilon^{g_i S - \widehat{g_i S}} = V^{\psi_\varepsilon * \alpha_i\, dx}$$

Then it is easy to see (using that $V = \Psi *$ and $|\Psi(x-y)| \leq \frac{A}{|x-y|^{n-1}}$) that

$$|U_\varepsilon^{\alpha_i}(z)| \leq A\,\|\psi_\varepsilon * \alpha_i\|_{L^\infty}\ \operatorname{diam}(\operatorname{supp}(\psi_\varepsilon * \alpha_i))\,. \tag{3.43}$$

But $\operatorname{diam}(\operatorname{supp}(\psi_\varepsilon * \alpha_i)) \asymp \varepsilon$ as $\varepsilon > \ell(Q_i)$.

On the other hand (we denote by x_i the center of Q_i),

$$\begin{aligned}(\psi_\varepsilon * \alpha_i)(w) &= \langle \psi_\varepsilon(w-\xi), \alpha_i(\xi)\rangle = \langle \psi_\varepsilon(w-\xi) - \psi_\varepsilon(w-x_i), \alpha_i(\xi)\rangle \\ &= \langle (\psi_\varepsilon(w-\xi) - \psi_\varepsilon(w-x_i))\eta_i(\xi), \alpha_i(\xi)\rangle\,,\end{aligned}$$

where $\eta_i \in C_0^\infty(3Q_i), \eta_i = 1$ on $2Q_i$ (recall that $\operatorname{supp}(\alpha_i) \subset 2Q_i$, $|\nabla \eta_i| \leq \frac{A}{\ell(Q_i)}$). We are almost in a position to use Lemma 3.2, namely (3.3) with $D = \frac{1}{\varepsilon^{n+1}}$. In fact, actually, Lemma 3.4 and relation (3.37) show that for a certain constant A (which depends only on the dimension), $\frac{1}{A}\Phi * \alpha_i \in L(E,1)$. So α_i plays the part of S in Lemma 3.2. To employ Lemma 3.2 we need to verify that $\varphi(\xi) := (\psi_\varepsilon(w-\xi) - \psi_\varepsilon(x_i - \xi))\eta_i(\xi)$ satisfies the assumptions of this lemma with $r \asymp \ell(Q_i), D = \frac{1}{\varepsilon^{n+1}}$. So let us verify that

$$\|\varphi\|_{L^\infty} \leq \frac{A\ell(Q_i)}{\varepsilon^{n+1}},\ \|\nabla\varphi\|_{L^\infty} \leq \frac{A}{\varepsilon^{n+1}},\ \|\nabla^2\varphi\|_{L^\infty} \leq \frac{A}{\ell(Q_i)\varepsilon^{n+1}}\,. \tag{3.44}$$

But it is easy to see that

$$\begin{aligned}|\varphi(\xi)| &\leq \frac{A}{\varepsilon^{n+1}}\|\psi\|_{L^\infty}\|\eta_i\|_{L^\infty} \leq \frac{A\ell(Q_i)}{\varepsilon^{n+1}}\,,\\ |\nabla\varphi(\xi)| &\leq |\psi_\varepsilon(w-\xi) - \psi_\varepsilon(w-x_i)||\nabla\eta_i(\xi)| + |\nabla\psi_\varepsilon(w-\xi)||\eta_i(\xi)| \\ &\leq \frac{A\ell(Q_i)}{\varepsilon^{n+1}}\|\nabla\psi\|_{L^\infty}\frac{A}{\ell(Q_i)} + \frac{A}{\varepsilon^{n+1}}\|\nabla\psi\|_{L^\infty}\|\eta_i\|_{L^\infty} \leq \frac{A}{\varepsilon^{n+1}}\,,\\ |\nabla^2\varphi(\xi)| &\leq \frac{A}{\varepsilon^{n+2}}\|\nabla^2\psi\|_{L^\infty}\|\eta_i\|_{L^\infty} + \frac{A}{\varepsilon^{n+1}}\|\nabla\psi\|_{L^\infty}\|\nabla\eta_i\|_{L^\infty} \\ &\quad + \frac{A\ell(Q_i)}{\varepsilon^{n+1}}\|\nabla\psi\|_{L^\infty}\|\nabla^2\eta_i\|_{L^\infty} \leq \frac{A}{\ell(Q_i)\varepsilon^{n+1}}\,.\end{aligned}$$

Finally we are in the position to use Lemma 3.2: $|\langle\varphi,\alpha_i\rangle| \leq \frac{A\,\ell(Q_i)\,\gamma(2Q_i\cap E)}{\varepsilon^{n+1}}$, or

$$\|\psi_\varepsilon * \alpha_i\|_{L^\infty} \leq \frac{A\,\ell(Q_i)\,\gamma(2Q_i \cap E)}{\varepsilon^{n+1}}\,. \tag{3.45}$$

We plug (3.45) into (3.43) to obtain

$$|U_\varepsilon^{\alpha_i}(z)| \le \frac{A\,\ell(Q_i)\,\gamma(2Q_i \cap E)}{\varepsilon^n} \le \frac{A\,\ell(Q_i)\,\gamma(2Q_i \cap E)}{\operatorname{dist}(z, 2Q_i)^n} \tag{3.46}$$

if $z \in \mathbb{R}^n \setminus 4Q_i$ and $\varepsilon > \frac{\operatorname{dist}(z,2Q_i)}{2}$. The combination of (3.42) and (3.46) finishes the proof of Lemma 3.7. □

Now, after the proof of this lemma, Theorem 3.6 is completely proved. It will serve as an important claim in the future: we will use it as an assumption of a certain Tb theorem which will prove the equivalence of γ and γ_+ in the case when $\gamma(E)$ is not much smaller than $\sum_{i=1}^N \gamma(2Q_i \cap E)$.