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CHAPTER 22

Two-Weight Hilbert Transform and Maximal Operator

22.1. Doubling

Let us introduce two maximal operators.

$$M_\mu f(x) := \sup_{I: x\in I} \frac{1}{|I|}\int_I |f|\,d\mu,\;\; M_\nu g(x) := \sup_{I: x\in I} \frac{1}{|I|}\int_I |g|\,d\nu\,.$$

By the works of E. Sawyer [**58**], [**59**], it is known when M_μ is a bounded operator from $L^2(\mu)$ to $L^2(\nu)$. This happens if and only if the uniform bound on test functions holds:

$$\|M_\mu \chi_I\|^2_\nu \le C_m\,\mu(I),\;\; \forall \text{ interval } I\,. \tag{22.1}$$

The symmetric condition (with exchanging μ and ν) is necessary and sufficient for the boundedness of M_ν:

$$\|M_\nu \chi_I\|^2_\mu \le C_m\,\nu(I),\;\; \forall \text{ interval } I\,. \tag{22.2}$$

Of course, (15.14), (15.15) of our Theorem (15.1):

$$\|H_\mu \chi_I\|^2_{L^2(\nu)} \le C_\chi \nu(I),\;\; \forall I\subset\mathbb{R}\,, \tag{22.3}$$

$$\|H_\nu \chi_I\|^2_{L^2(\mu)} \le C_\chi \mu(I),\;\; \forall I\subset\mathbb{R}\,. \tag{22.4}$$

are the analogs of these Sawyer conditions, but applied to a singular operator.

The drawback of our main result about the two–weight Hilbert transform is that we were obliged to assume the "smoothness" of measures to get the necessary and sufficient conditions for the boundedness of $H_\mu : L^2(\mu)\to L^2(\nu)$. Smoothness meant that the support of the measure is the whole line (or circle) and the doubling property holds.

The next theorem does not have the smoothness requirement but it has still the doubling requirement. It is now understood as

$$\mu(2I)\le C_d\mu(I),\text{ if } \mu(I)>0,\;\; \nu(2I)\le C_d\nu(I),\text{ if } \nu(I)>0\,. \tag{22.5}$$

The theorem below gives the necessary and sufficient conditions not for the boundedness of $H_\mu : L^2(\mu)\to L^2(\nu)$, but for the boundedness of the *family* consisting of three operators: H_μ, M_μ, M_ν. Such *family* results can be found in the literature. See, for example, [**50**], where the two–weight criterion has been found for the family of all Martingale Transforms.

THEOREM 22.1. *If operators H_μ, M_μ are bounded from $L^2(\mu)$ to $L^2(\nu)$, and M_ν is bounded from $L^2(\nu)$ to $L^2(\mu)$, then the constants C_m, C_χ in* (22.1), (22.2), (22.3), (22.4)*, and the constant C_p from* (15.16) *are finite and bounded by* $A\,(\|H_\mu\|_{L^2(\mu)\to L^2(\nu)} + \|M_\mu\|_{L^2(\mu)\to L^2(\nu)} + \|M_\nu\|_{L^2(\nu)\to L^2(\mu)})^2$.

On the other hand, let (22.5) *be satisfied; then the norms of these three operators are bounded by* $C(C_d, C_m, C_\chi, C_p) < \infty$*, if the constants* C_d*,* C_m*,* C_χ*,* C_p *are finite. In other words, the family* H_μ, M_μ, M_ν *consists of bounded operators if and only if test conditions* (22.1)*,* (22.2)*,* (22.3)*,* (22.4)*, and* (15.16) *are satisfied.*

Remark. It only seems that the theorem is asymmetric. Of course, the boundedness of H_μ from $L^2(\mu)$ to $L^2(\nu)$ is the same as the boundedness of minus its adjoint H_ν from $L^2(\nu)$ to $L^2(\mu)$. Note also that the essence of the theorem is that the norm $\|H_\mu\|_{L^2(\mu)\to L^2(\nu)}$ can be bounded by $C(C_d, C_m, C_\chi, C_p)$. The rest is either obvious or follows from Sawyer's two–weight result for maximal functions cited above.

Proof. We repeat verbatim all lines of the proof of Theorem 15.1. Again introduce good and bad functions, and reduce by averaging over probability the estimate of H_μ to its estimate on good functions. Everything goes without change up to the moment when we have to prove Theorem 20.1. There the smoothness was used; we do not have it now.

So here are the changes in the proof of Theorem 15.1. First of all, we change a bit the stopping criterion (20.7). Now, given $\hat{I} \in \mathcal{D}^\mu$, we (fixing a large K) consider the largest $I \subset \hat{I}, I \in \mathcal{D}^\mu$ such that

$$\left[P_I(\chi_{\hat{I}}\, d\mu)\right]^2 \nu(I) \geq K\, \mu(I), \;\; \mu(I) > 0\,. \tag{22.6}$$

Call the family of such I's by $\mathcal{F}(\hat{I})$. Then we call the interval $S \subset \hat{I}, S \in \mathcal{D}^\mu$ *a stopping interval* if either it is a grandfather of such an $I \in \mathcal{F}(\hat{I})$ or $\mu(S) = 0$.

Here is the place, where we use the finiteness of C_m instead of the smoothness of measure property:

LEMMA 22.2. *If* C_m, C_d *are finite, then for every* $\hat{I} \in \mathcal{D}^\mu$*,*

$$\sum_{S\in\mathcal{D}^\mu,\, S\subset\hat{I},\, S\,\text{is maximal stopping}} \mu(S) \leq \frac{1}{2}\mu(\hat{I})\,, \tag{22.7}$$

provided that the constant K *in the stopping criterion* (22.6) *is large enough.*

Proof. It is a standard estimate of the Poisson integral via the maximal function (see, for example, [**21**]), which gives

$$P_I(\chi_{\hat{I}}\, d\mu) \leq A\, \inf_{x\in I} (M_\mu \chi_{\hat{I}})(x)\,. \tag{22.8}$$

Then (22.8) implies

$$\begin{aligned} K \sum_{I:\, I\in\mathcal{F}(\hat{I}), I\subset\hat{I}} \mu(I) &\leq \sum_{I:\, I\in\mathcal{F}(\hat{I}), I\subset\hat{I}} P_I(\chi_{\hat{I}}\, d\mu)^2 \nu(I) \\ &\leq \int_{\cup I\; I\in\mathcal{F}(\hat{I}), I\subset\hat{I}} (M_\mu\chi_{\hat{I}})(x)^2\, d\nu(x) \leq \int (M_\mu\chi_{\hat{I}})(x)^2\, d\nu(x) \leq C_m \mu(\hat{I})\,. \end{aligned}$$

If S is a maximal stopping interval inside $\hat{I}$, then either $\mu(S) = 0$ or it contains a grandson $I \in \mathcal{F}(\hat{I})$ of positive measure. Use (22.5). Then (22.7) is proved if K is larger than $A\, C_d\, C_m$. □

We construct stopping intervals of the next generation inside stopping intervals already constructed, and we continue this process. We obtained the family of all stopping intervals of all generations. Call it $\mathcal{S}$ as before.

Let $\hat{I}$ be a stopping interval, $\mu(\hat{I}) > 0$, and let $I \subset \hat{I}, I \in \mathcal{D}^\mu$ be such that $\hat{I}$ is the smallest interval from $\mathcal{S}$ containing I; then, without loss of generality, we can consider only the case $\mu(I) > 0$ (otherwise any $f \in L^2(\mu)$ does not live on I, and we do not need to consider $(\Delta_I^\mu f, \dots)_\nu$ for $I \subset \hat{I}$). Let now $I_i, i = 1, 2$ be two halves of I; then, by construction,

$$\Big[P_{I_i}(\chi_{\hat{I}\setminus I_i}\, d\mu)\Big]^2 \nu(I) \le K\, \mu(I_i), \;\; i = 1, 2\,. \tag{22.9}$$

The relation (22.9) is the only one that was used in the estimate of stopping terms.

The estimate of paraproducts is verbatim the same as in Section 21.2. Again we have the right to consider only those $S \in \mathcal{S}$ for which $\mu(S) > 0$. For such S, one has

$$\Big[P_S(\chi_{\hat{S}\setminus S}\, d\mu)\Big]^2 \nu(S) \le K\, \mu(S)\,, \tag{22.10}$$

and this is the only thing we used for the estimates of paraproducts. □

22.2. No doubling

THEOREM 22.3. *If operators H_μ, M_μ are bounded from $L^2(\mu)$ to $L^2(\nu)$, and M_ν is bounded from $L^2(\nu)$ to $L^2(\mu)$, then the constants C_m, C_χ in* (22.1), (22.2), (22.3), (22.4), *and the constant C_p from* (15.16) *are finite and bounded by $A\,(\|H_\mu\|_{L^2(\mu)\to L^2(\nu)} + \|M_\mu\|_{L^2(\mu)\to L^2(\nu)} + \|M_\nu\|_{L^2(\nu)\to L^2(\mu)})^2$.*

On the other hand, the norms of these three operators are bounded by constants depending on C_m, C_χ, C_p, if the constants C_m, C_χ, C_p are finite. In other words, the family H_μ, M_μ, M_ν consists of bounded operators if and only if test conditions (22.1), (22.2), (22.3), *and* (22.4) *are satisfied.*

The difference with Theorem 22.1 is in the absence of the doubling assumption on the measures. The proof is slightly more complicated because the stopping time should be chosen more carefully. We will provide the proof elsewhere.