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Ken Ono

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Preface

Modular forms appear in many ways in number theory. They play a central role in the theory of quadratic forms; in particular, they are generating functions for the number of representations of integers by positive definite quadratic forms (for example, see [**Gro**]). They are also key players in the recent spectacular proof of Fermat's Last Theorem (see for example, [**Bos, CSS**]). Modular forms are presently at the center of an immense amount of research activity.

Here I describe other roles that modular forms and q-series play in number theory. Sarnak's elegant monograph [**Sar**] describes the implications of bounding Fourier coefficients of modular forms. Recent books such as [**Bos, CSS, Hi2**] focus on questions involving Galois representations associated to modular forms and questions in arithmetical algebraic geometry. I focus on complementary issues involving the arithmetic and combinatorics of modular forms, and their further roles in number theory.

The vast amount of research currently being done makes it impossible to provide a comprehensive account in eleven chapters. Obviously, the subjects I choose to emphasize are a matter of personal taste, and they reflect the projects which have most actively engaged my efforts. Here we consider recent work on partitions, basic hypergeometric functions, Gaussian hypergeometric functions, super-congruences, Weierstrass points on modular curves, singular moduli, class numbers, L-values, and elliptic curves. This monograph is an expanded version of the ten lectures I presented at the NSF-CBMS Regional Conference at the University of Illinois at Urbana-Champaign. The ten lectures were:

- Preview.
- Facts and tools.
- Infinite products expansions of modular forms.
- Congruences for partitions and singular moduli.
- Supersingular j-invariants.
- Class numbers of quadratic fields.
- Nonvanishing of L-functions.
- Basic hypergeometric series and L-values.
- Gaussian hypergeometric functions and modular forms.
- Open Problems.

I assume some familiarity with modular forms, although many of the basic facts are given here without proof. There are many introductory textbooks which are useful references such as [**Apo, Hi1, Iw2, Kna, Kno, Kob2, La2, Mi2, Ogg1, Ran, Sch, Se2, Shi1**]. Thanks to the efforts of Stein and others, there are now computer packages such as Magma which can carry out modular form calculations which were inconceivable a few years ago. These new resources are also invaluable.

In this monograph I hope to provide ample motivation on some of the topics in which modular forms play a role. I do not make an attempt to catalog all of the results in these areas; rather I highlight results which give their flavor. At the end of most chapters, I list some open problems and questions. I invite solutions, and I look forward to substantial progress on these topics.

I owe a great debt to many people who were involved with the NSF-CBMS Regional Conference at the University of Illinois at Urbana-Champaign, where these lectures were presented. Scott Ahlgren and Bruce Berndt organized a wonderful conference which ran very smoothly. I also thank Jan Bruinier, YoungJu Choie, Masanobu Kaneko, Winfried Kohnen, and Ram Murty, the invited speakers, who delivered exciting lectures on many of the topics described here. I would also like to thank Scott Ahlgren, Matt Boylan, Jan Bruinier, Denis Charles, Rohit Chatterjee, Ahmad El-Guindy, Jayce Getz, Masanobu Kaneko, Winfried Kohnen, Jeremy Lovejoy, Karl Mahlburg, Bill McGraw, Jeremy Rouse, and Kathryn Zuhr for their many helpful comments on early versions of this manuscript. I also thank my wife Erika, and my children, Aspen and Sage, for their patience and tremendous support. Without them this project would not have been possible. To all of these people, my sincerest thanks.

Ken Ono
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Modular forms appear in many ways in number theory. They play a central role in the theory of quadratic forms; in particular, they are generating functions for the number of representations of integers by positive definite quadratic forms. They are also key players in the recent spectacular proof of Fermat's Last Theorem. Modular forms are currently at the center of an immense amount of research activity. Other roles that modular forms and q-series play in number theory are described in this book. In particular, applications and connections to basic hypergeometric functions, Gaussian hypergeometric functions, super-congruences, Weierstrass points on modular curves, singular moduli, class numbers, L-values, and elliptic curves are described in detail.

The first three chapters of the book provide some basic facts and results on modular forms, setting the stage for the remainder of the book, where advanced topics are treated. Ono provides ample motivation on some of the topics in which modular forms play a role. There is no attempt to catalog all of the results in these areas; rather, the author highlights results which give their flavor. At the end of most chapters, there are some open problems and questions.

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