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Wave Packet Analysis

Christoph Thiele



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Preface

These notes arose from a series of lectures I gave at the NSF/CBMS Regional Conference in the Mathematical Sciences at the Georgia Institute of Technology May 23-28 2004. The other speakers at that conference were A. Iosevich, I. Laba, X. Li, A. Magyar, C. Muscalu, K. Oskolkov, A. Seeger, all of whom I would like to thank, as well as all attendees of this conference.

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Christoph Thiele

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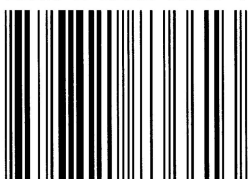
Wave Packet Analysis

The concept of “wave packet analysis” originates in Carleson’s famous proof of almost everywhere convergence of Fourier series of L^2 functions. It was later used by Lacey and Thiele to prove bounds on the bilinear Hilbert transform. For quite some time, Carleson’s wave packet analysis was thought to be an important idea, but that it had limited applications. But in recent years, it has become clear that this is an important tool for a number of other applications. This book is an introduction to these tools. It emphasizes the classical successes (Carleson’s theorem and the Hilbert transform) in the main development. However, the book closes with a dedicated chapter on more recent results.

Carleson’s original theorem is sometimes cited as one of the most important developments of 20th century harmonic analysis. The set of ideas stemming from his proof is now seen as an essential element in modern harmonic analysis. Indeed, Thiele won the Salem prize jointly with Michael Lacey for work in this area.

The book gives a nice survey of important material, such as an overview of the theory of singular integrals and wave packet analysis itself. There is a separate chapter on “further developments”, which gives a broader view on the subject, though it does not exhaust all ongoing developments.

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