Nonlinear Dispersive Equations
Local and Global Analysis
Terence Tao
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Nonlinear Dispersive Equations
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Terence Tao
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To Laura, for being so patient.
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Preface

*Politics is for the present, but an equation is something for eternity.*
(Albert Einstein)

This monograph is based on (and greatly expanded from) a lecture series given at the NSF-CBMS regional conference on nonlinear and dispersive wave equations at New Mexico State University, held in June 2005. Its objective is to present some aspects of the global existence theory (and in particular, the regularity and scattering theory) for various nonlinear dispersive and wave equations, such as the Korteweg-de Vries (KdV), nonlinear Schrödinger, nonlinear wave, and wave maps equations. The theory here is rich and vast and we cannot hope to present a comprehensive survey of the field here; our aim is instead to present a sample of results, and to give some idea of the motivation and general philosophy underlying the problems and results in the field, rather than to focus on the technical details. We intend this monograph to be an introduction to the field rather than an advanced text; while we do include some very recent results, and we imbue some more classical results with a modern perspective, our main concern will be to develop the fundamental tools, concepts, and intuitions in as simple and as self-contained a matter as possible. This is also a pedagogical text rather than a reference; many details of arguments are left to exercises or to citations, or are sketched informally. Thus this text should be viewed as being complementary to the research literature on these topics, rather than being a substitute for them.

The analysis of PDE is a beautiful subject, combining the rigour and technique of modern analysis and geometry with the very concrete real-world intuition of physics and other sciences. Unfortunately, in some presentations of the subject (at least in pure mathematics), the former can obscure the latter, giving the impression of a fearsomely technical and difficult field to work in. To try to combat this, this book is devoted in equal parts to rigour and to intuition; the usual formalism of definitions, propositions, theorems, and proofs appear here, but will be interspersed and complemented with many informal discussions of the same material, centering around vague “Principles” and figures, appeal to physical intuition and examples, back-of-the-envelope computations, and even some whimsical quotations. Indeed, the exposition and exercises here reflect my personal philosophy that to truly understand a mathematical result one must view it from as many perspectives as possible (including both rigorous arguments and informal heuristics), and must also be able to translate easily from one perspective to another. The reader should thus be aware of which statements in the text are rigorous, and which ones are heuristic, but this should be clear from context in most cases.

To restrict the field of study, we shall focus primarily on defocusing equations, in which soliton-type behaviour is prohibited. From the point of view of global existence, this is a substantially easier case to study than the focusing problem, in
which one has the fascinating theory of solitons and multi-solitons, as well as various mechanisms to enforce blow-up of solutions in finite or infinite time. However, we shall see that there are still several analytical subtleties in the defocusing case, especially when considering critical nonlinearities, or when trying to establish a satisfactory scattering theory. We shall also work in very simple domains such as Euclidean space $\mathbb{R}^n$ or tori $\mathbb{T}^n$, thus avoiding consideration of boundary-value problems, or curved space, though these are certainly very important extensions to the theory. One further restriction we shall make is to focus attention on the initial value problem when the initial data lies in a Sobolev space $H^s_0(\mathbb{R}^d)$, as opposed to more localised choices of initial data (e.g. in weighted Sobolev spaces, or self-similar initial data). This restriction, combined with the previous one, makes our choice of problem translation-invariant in space, which leads naturally to the deployment of the Fourier transform, which turns out to be a very powerful tool in our analysis. Finally, we shall focus primarily on only four equations: the semilinear Schrödinger equation, the semilinear wave equation, the Korteweg-de Vries equation, and the wave maps equation. These four equations are of course only a very small sample of the nonlinear dispersive equations studied in the literature, but they are reasonably representative in that they showcase many of the techniques used for more general equations in a comparatively simple setting.

Each chapter of the monograph is devoted to a different class of differential equations; generally speaking, in each chapter we first study the algebraic structure of these equations (e.g. symmetries, conservation laws, and explicit solutions), and then turn to the analytic theory (e.g. existence and uniqueness, and asymptotic behaviour). The first chapter is devoted entirely to *ordinary differential equations* (ODE). One can view partial differential equations (PDE) such as the nonlinear dispersive and wave equations studied here, as infinite-dimensional analogues of ODE; thus finite-dimensional ODE can serve as a simplified model for understanding techniques and phenomena in PDE. In particular, basic PDE techniques such as Picard and Duhamel iteration, energy methods, continuity or bootstrap arguments, conservation laws, near-conservation laws, and monotonicity formulae all have useful ODE analogues. Furthermore, the analogy between classical mechanics and quantum mechanics provides a useful heuristic correspondence between Schrödinger type equations, and classical ODE involving one or more particles, at least in the high-frequency regime.

The second chapter is devoted to the theory of the basic linear dispersive models: the *Airy equation*, the *free Schrödinger equation*, and the *free wave equation*. In particular, we show how the Fourier transform and conservation law methods, can be used to establish existence of solutions, as well as basic estimates such as the dispersive estimate, local smoothing estimates, Strichartz estimates, and $X^{s,b}$ estimates.

In the third chapter we begin studying nonlinear dispersive equations in earnest, beginning with two particularly simple semilinear models, namely the *nonlinear Schrödinger equation* (NLS) and *nonlinear wave equation* (NLW). Using these equations as examples, we illustrate the basic approaches towards defining and constructing solutions, and establishing local and global properties, though we defer the study of the more delicate energy-critical equations to a later chapter. (The mass-critical nonlinear Schrödinger equation is also of great interest, but we will not discuss it in detail here.)
In the fourth chapter, we analyze the *Korteweg de Vries equation* (KdV), which requires some more delicate analysis due to the presence of derivatives in the nonlinearity. To partly compensate for this, however, one now has the structures of nonresonance and complete integrability; the interplay between the integrability on one hand, and the Fourier-analytic structure (such as nonresonance) on the other, is still only partly understood, however we are able to at least establish a quite satisfactory local and global wellposedness theory, even at very low regularities, by combining methods from both. We also discuss a less dispersive cousin of the KdV equation, namely the *Benjamin-Ono equation*, which requires more nonlinear techniques, such as gauge transforms, in order to obtain a satisfactory existence and wellposedness theory.

In the fifth chapter we return to the semilinear equations (NLS and NLW), and now establish large data global existence for these equations in the defocusing, energy-critical case. This requires the full power of the local wellposedness and perturbation theory, together with Morawetz-type estimates to prevent various kinds of energy concentration. The situation is especially delicate for the Schrödinger equation, in which one must employ the induction on energy methods of Bourgain in order to obtain enough structural control on a putative *minimal energy blowup solution* to obtain a contradiction and thus ensure global existence.

In the final chapter, we turn to the *wave maps equation* (WM), which is somewhat more nonlinear than the preceding equations, but which on the other hand enjoys a strongly geometric structure, which can in fact be used to renormalise most of the nonlinearity. The small data theory here has recently been completed, but the large data theory has just begun; it appears however that the geometric renormalisation provided by the harmonic map heat flow, together with a Morawetz estimate, can again establish global existence in the negatively curved case.

As a final disclaimer, this monograph is by no means intended to be a definitive, exhaustive, or balanced survey of the field. Somewhat unavoidably, the text focuses on those techniques and results which the author is most familiar with, in particular the use of the iteration method in various function spaces to establish a local and perturbative theory, combined with frequency analysis, conservation laws, and monotonicity formulae to then obtain a global non-perturbative theory. There are other approaches to this subject, such as via compactness methods, nonlinear geometric optics, infinite-dimensional Hamiltonian dynamics, or the techniques of complete integrability, which are also of major importance in the field (and can sometimes be combined, to good effect, with the methods discussed here); however, we will be unable to devote a full-length treatment of these methods in this text. It should also be emphasised that the methods, heuristics, principles and philosophy given here are tailored for the goal of analyzing the Cauchy problem for semilinear dispersive PDE; they do not necessarily extend well to other PDE questions (e.g. control theory or inverse problems), or to other classes of PDE (e.g. conservation laws or to parabolic and elliptic equations), though there are certain many connections and analogies between results in dispersive equations and in other classes of PDE.

I am indebted to my fellow members of the "I-team" (Jim Colliander, Markus Keel, Gigliola Staffilani, Hideo Takaoka), to Sergiu Klainerman, and to Michael Christ for many entertaining mathematical discussions, which have generated much of the intuition that I have tried to place into this monograph. I am also very
thankful for Jim Ralston for using this text to teach a joint PDE course, and providing me with careful corrections and other feedback on the material. I also thank Soonsik Kwon for additional corrections. Last, but not least, I am grateful to my wife Laura for her support, and for pointing out the analogy between the analysis of nonlinear PDE and the electrical engineering problem of controlling feedback, which has greatly influenced my perspective on these problems (and has also inspired many of the diagrams in this text).

Terence Tao

Notation. As is common with any book attempting to survey a wide range of results by different authors from different fields, the selection of a unified notation becomes very painful, and some compromises are necessary. In this text I have (perhaps unwisely) decided to make the notation as globally consistent across chapters as possible, which means that any individual result presented here will likely have a notation slightly different from the way it is usually presented in the literature, and also that the notation is more finicky than a local notation would be (often because of some ambiguity that needed to be clarified elsewhere in the text). For the most part, changing from one convention to another is a matter of permuting various numerical constants such as 2, π, i, and -1; these constants are usually quite harmless (except for the sign -1), but one should nevertheless take care in transporting an identity or formula in this book to another context in which the conventions are slightly different.

In this text, d will always denote the dimension of the ambient physical space, which will either be a Euclidean space \( \mathbb{R}^d \) or the torus \( \mathbb{T}^d := (\mathbb{R}/2\pi \mathbb{Z})^d \). (Chapter 1 deals with ODE, which can be considered to be the case \( d = 0 \).) All integrals on these spaces will be with respect to Lebesgue measure \( dx \). If \( x = (x_1, \ldots, x_d) \) and \( \xi = (\xi_1, \ldots, \xi_d) \) lie in \( \mathbb{R}^d \), we use \( x \cdot \xi \) to denote the dot product \( x \cdot \xi := x_1 \xi_1 + \ldots + x_d \xi_d \), and \( |x| \) to denote the magnitude \( |x| := (x_1^2 + \ldots + x_d^2)^{1/2} \). We also use \( \langle x \rangle \) to denote the inhomogeneous magnitude (or Japanese bracket) \( \langle x \rangle := (1 + |x|^2)^{1/2} \) of \( x \), thus \( \langle x \rangle \) is comparable to \( |x| \) for large \( x \) and comparable to 1 for small \( x \). In a similar spirit, if \( x = (x_1, \ldots, x_d) \in \mathbb{T}^d \) and \( k = (k_1, \ldots, k_d) \in \mathbb{Z}^d \) we define \( k \cdot x := k_1 x_1 + \ldots + k_d x_d \in \mathbb{T} \). In particular the quantity \( e^{i k \cdot x} \) is well-defined.

We say that \( I \) is a time interval if it is a connected subset of \( \mathbb{R} \) which contains at least two points (so we allow time intervals to be open or closed, bounded or unbounded). If \( u : I \times \mathbb{R}^d \to \mathbb{C}^n \) is a (possibly vector-valued) function of spacetime, we write \( \partial_t u \) for the time derivative \( \frac{\partial u}{\partial t} \), and \( \partial_x u, \ldots, \partial_{x_d} u \) for the spatial derivatives \( \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_d} \); these derivatives will either be interpreted in the classical sense (when \( u \) is smooth) or the distributional (weak) sense (when \( u \) is rough). We use \( \nabla_x u : I \times \mathbb{R}^d \to \mathbb{C}^{n \times d} \) to denote the spatial gradient \( \nabla_x u = (\partial_{x_1} u, \ldots, \partial_{x_d} u) \). We can iterate this gradient to define higher derivatives \( \nabla_x^k u \) for \( k = 0, 1, \ldots \) Of course,

\(^1\)We will be using two slightly different notions of spacetime, namely Minkowski space \( \mathbb{R}^{1+d} \) and Galilean spacetime \( \mathbb{R} \times \mathbb{R}^d \); in the very last section we also need to use parabolic spacetime \( \mathbb{R}^+ \times \mathbb{R}^d \). As vector spaces, they are of course equivalent to each other (and to the Euclidean space \( \mathbb{R}^{d+1} \)), but we will place different (pseudo)metric structures on them. Generally speaking, wave equations will use Minkowski space, whereas nonrelativistic equations such as Schrödinger equations will use Galilean spacetime, while heat equations use parabolic spacetime. For the most part the reader will be able to safely ignore these subtle distinctions.
these definitions also apply to functions on \( \mathbb{T}^d \), which can be identified with periodic functions on \( \mathbb{R}^d \).

We use the Einstein convention for summing indices, with Latin indices ranging from 1 to \( d \), thus for instance \( x_j \partial_{x_j} u \) is short for \( \sum_{j=1}^d x_j \partial_{x_j} u \). When we come to wave equations, we will also be working in a Minkowski space \( \mathbb{R}^{1+d} \) with a Minkowski metric \( g_{\alpha\beta} \); in such cases, we will use Greek indices and sum from 0 to \( d \) (with \( x^0 = t \) being the time variable), and use the metric to raise and lower indices. Thus for instance if we use the standard Minkowski metric \( dg^2 = -dt^2 + |dx|^2 \), then \( \partial_0 u = \partial_t u \) but \( \partial^0 u = -\partial_t u \).

In this monograph we always adopt the convention that \( \int_s^t = -\int_t^s \) if \( t < s \). This convention will usually be applied only to integrals in the time variable.

We use the Lebesgue norms

\[
\|f\|_{L^p_x(\mathbb{R}^d \to \mathbb{C})} := (\int_{\mathbb{R}^d} |f(x)|^p \, dx)^{1/p}
\]

for \( 1 \leq p < \infty \) for complex-valued measurable functions \( f : \mathbb{R}^d \to \mathbb{C} \), with the usual convention

\[
\|f\|_{L^\infty_x(\mathbb{R}^d \to \mathbb{C})} := \text{ess sup}_{x \in \mathbb{R}^d} |f(x)|.
\]

In many cases we shall abbreviate \( L^p_x(\mathbb{R}^d \to \mathbb{C}) \) as \( L^p(\mathbb{R}^d) \), \( L^p(\mathbb{R}^d) \), or even \( L^p \) when there is no chance of confusion. The subscript \( x \), which denotes the dummy variable, is of course redundant. However we will often retain it for clarity when dealing with PDE, since in that context one often needs to distinguish between Lebesgue norms in space \( x \), time \( t \), spatial frequency \( \xi \), or temporal frequency \( \tau \). Also we will need it to clarify expressions such as \( \|xf\|_{L^p_x(\mathbb{R}^d)} \), in which the expression in the norm depends explicitly on the variable of integration. We of course identify any two functions if they agree almost everywhere. One can of course replace the domain \( \mathbb{R}^d \) by the torus \( \mathbb{T}^d \) or the lattice \( \mathbb{Z}^d \), thus for instance

\[
\|f\|_{L^p_x(\mathbb{Z}^d \to \mathbb{C})} := (\sum_{k \in \mathbb{Z}^d} |f(k)|^p)^{1/p}.
\]

One can replace \( \mathbb{C} \) by any other Banach space \( X \), thus for instance \( L^p_x(\mathbb{R}^d \to X) \) is the space of all measurable functions \( u : \mathbb{R}^d \to X \) with finite norm

\[
\|u\|_{L^p_x(\mathbb{R}^d \to X)} := (\int_{\mathbb{R}^d} \|u(x)\|^p_X \, dx)^{1/p}
\]

with the usual modification for \( p = \infty \). In particular we can define the mixed Lebesgue norms \( L^p_I L^q_x(\mathbb{R}^d) \) for any time interval \( I \) as the space of all functions \( u : I \times \mathbb{R}^d \to \mathbb{C} \) with norm

\[
\|u\|_{L^p_I L^q_x(\mathbb{R}^d \to \mathbb{C})} := (\int_I \|u(t)\|_{L^q_x(\mathbb{R}^d)}^q \, dt)^{1/q} = (\int_{\mathbb{R}^d} (\int_I |u(t,x)|^r \, dt)^{q/r} \, dx)^{1/q}
\]

with the usual modifications when \( q = \infty \) or \( r = \infty \). One can also use this Banach space notation to make sense of the \( L^p \) norms of tensors such as \( \nabla f \), \( \nabla^2 f \), etc., provided of course that such derivatives exist in the \( L^p \) sense.

In a similar spirit, if \( I \) is a time interval and \( k \geq 0 \), we use \( C^k_I(\mathbb{R}^d) \) to denote the space of all \( k \)-times continuously differentiable functions \( u : I \to \mathbb{C} \).
the norm
\[ \|u\|_{C^k(I \to X)} := \sum_{j=0}^k \|\partial_j^k u\|_{L^\infty(I \to X)}. \]

We adopt the convention that \( \|u\|_{C^k(I \to X)} = \infty \) if \( u \) is not \( k \)-times continuously differentiable. One can of course also define spatial analogues \( C^k_t(x) \) of these spaces, as well as spacetime versions \( C^k_{t,x}(I \times \mathbb{R}^d \to X) \). We caution that if \( I \) is not compact, then it is possible for a function to be \( k \)-times continuously differentiable but have infinite \( C^k_t \) norm; in such cases we say that \( u \in C^k_{t,\text{loc}}(I \to X) \) rather than \( u \in C^k_t(I \to X) \). More generally, a statement of the form \( u \in X_{\text{loc}}(\Omega) \) on a domain \( \Omega \) means that we can cover \( \Omega \) by open sets \( V \) such that the restriction \( u|_V \) of \( u \) to each of these sets \( V \) is in \( X(\Omega) \); under reasonable assumptions on \( X \), this also implies that \( u|K \in X(K) \) for any compact subset \( K \) of \( \Omega \). As a rule of thumb, the global spaces \( X(\Omega) \) will be used for quantitative control (estimates), whereas the local spaces \( X_{\text{loc}}(\Omega) \) are used for qualitative control (regularity); indeed, the local spaces \( X_{\text{loc}} \) are typically only Frechet spaces rather than Banach spaces. We will need both types of control in this text, as one typically needs qualitative control to ensure that the quantitative arguments are rigorous.

If \( (X,d_X) \) is a metric space and \( Y \) is a Banach space, we use \( \dot{C}^{0,1}(X \to Y) \) to denote the space of all Lipschitz continuous functions \( f : X \to Y \), with norm
\[ \| f \|_{\dot{C}^{0,1}(X \to Y)} := \sup_{x,x' \in X : x \neq x'} \frac{\| f(x) - f(x') \|_Y}{d_X(x,x')}. \]

(One can also define the inhomogeneous Lipschitz norm \( \| f \|_{\dot{C}^{0,1}} := \| f \|_{\dot{C}^{0,1}} + \| f \|_{\dot{C}^0} \), but we will not need this here.) Thus for instance \( \dot{C}^1(\mathbb{R}^d \to \mathbb{R}^m) \) is a subset of \( \dot{C}^{0,1}(\mathbb{R}^d \to \mathbb{R}^m) \), which is in turn a subset of \( C^0_{\text{loc}}(\mathbb{R}^d \to \mathbb{R}^m) \). The space \( \dot{C}^{0,1}_{\text{loc}}(X \to Y) \) is thus the space of locally Lipschitz functions (i.e. every \( x \in X \) is contained in a neighbourhood on which the function is Lipschitz).

In addition to the above function spaces, we shall also use Sobolev spaces \( H^s, W^{s,p}, H^s, W^{s,p} \), which are defined in Appendix A, and \( X^{s,b} \) spaces, which are defined in Section 2.6.

If \( V \) and \( W \) are finite-dimensional vector spaces, we use \( \text{End}(V \to W) \) to denote the space of linear transformations from \( V \) to \( W \), and \( \text{End}(V) = \text{End}(V \to V) \) for the ring of linear transformations from \( V \) to itself. This ring contains the identity transformation \( \text{id} = \text{id}_V \).

If \( X \) and \( Y \) are two quantities (typically non-negative), we use \( X \lesssim Y \) or \( Y \gtrsim X \) to denote the statement that \( X \leq CY \) for some absolute constant \( C > 0 \). We use \( X = O(Y) \) synonymously with \( |X| \lesssim Y \). More generally, given some parameters \( a_1, \ldots, a_k \), we use \( X \lesssim_{a_1, \ldots, a_k} Y \) or \( Y \gtrsim_{a_1, \ldots, a_k} X \) to denote the statement that \( X \leq C_{a_1, \ldots, a_k} Y \) for some (typically large) constant \( C_{a_1, \ldots, a_k} > 0 \) which can depend on the parameters \( a_1, \ldots, a_k \), and define \( X = O_{a_1, \ldots, a_k}(Y) \) similarly. Typical choices of parameters include the dimension \( d \), the regularity \( s \), and the exponent \( p \). We will also say that \( X \) is controlled by \( a_1, \ldots, a_k \) if \( X = O_{a_1, \ldots, a_k}(1) \). We use \( X \sim Y \) to denote the statement \( X \lesssim Y \lesssim X \), and similarly \( X \sim_{a_1, \ldots, a_k} Y \) denotes \( X \lesssim_{a_1, \ldots, a_k} Y \lesssim_{a_1, \ldots, a_k} X \). We will occasionally use the notation \( X \ll_{a_1, \ldots, a_k} Y \) or \( Y \gg_{a_1, \ldots, a_k} X \) to denote the statement \( X \leq c_{a_1, \ldots, a_k} Y \) for some suitably small quantity \( c_{a_1, \ldots, a_k} > 0 \) depending on the parameters \( a_1, \ldots, a_k \). This notation is...
somewhat imprecise (as one has to specify what “suitably small” means) and so we shall usually only use it in informal discussions.

Recall that a function $f : \mathbb{R}^d \to \mathbb{C}$ is said to be rapidly decreasing if we have
\[
\|\langle x \rangle^N f(x)\|_{L_2^\infty(\mathbb{R}^d)} < \infty
\]
for all $N \geq 0$. We then say that a function is Schwartz if it is smooth and all of its derivatives $\partial_x^\alpha f$ are rapidly decreasing, where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d$ ranges over all multi-indices, and $\partial_x^\alpha$ is the differential operator
\[
\partial_x^\alpha := \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial x_d} \right)^{\alpha_d}.
\]
In other words, $f$ is Schwartz if and only if $\partial_x^\alpha f(x) = O_{f,\alpha,N}(\langle x \rangle^{-N})$ for all $\alpha \in \mathbb{Z}_+^d$ and all $N$. We use $\mathcal{S}(\mathbb{R}^d)$ to denote the space of all Schwartz functions. As is well known, this is a Frechet space, and thus has a dual $\mathcal{S}'(\mathbb{R}^d)^*$, the space of tempered distributions. This space contains all locally integrable functions of polynomial growth, and is also closed under differentiation, multiplication with functions $g$ of symbol type (i.e. $g$ and all its derivatives are of polynomial growth) and convolution with Schwartz functions; we will not present a detailed description of the distributional calculus here.
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