http://dx.doi.org/10.1090/cbms/111/06

CHAPTER 6

The Topology of T-Duality and the Bunke-Schick Construction

6.1. Topological T-Duality

6.1.1. A Key Example.

6.1.1.1. *Topology Change in T-Duality.* We talked about the simplest case of T-duality in Section 1.3.2.1. There, string theory on $X = Z \times T$, where $T = S^1$ is a circle of radius R, corresponds to a dual string theory on $X^\sharp = Z \times T^\sharp$, where $T^\sharp$ is the dual circle with radius $\widetilde{R} = \frac{\alpha'}{R}$. But what if X is fibered by circles, but doesn't split as a product?

The first example of this phenomenon was studied by Alvarez, Alvarez-Gaumé, Barbón, and Lozano in [**3**]. (However, the reader should be cautioned that this paper did not get all of the details exactly right.) Their discovery was generalized 10 years later by Bouwknegt, Evslin, and Mathai in [**22, 23**]. Let's start with the simplest example of a circle fibration, where $X = S^3$, identified with $SU(2)$, and T ($\cong S^1$) is a maximal torus. Then T acts freely on X (say by right translation) and the quotient X/T is $\mathbb{CP}^1 \cong S^2$, with quotient map $p\colon X \to S^2$ the *Hopf fibration*. Assume for simplicity that the B-field on X vanishes.

Let's examine this case in more detail. We have $X = S^3$ fibering over $Z = X/T = S^2$. Think of Z as the union of the two hemispheres $Z^\pm \cong D^2$ intersecting in the equator $Z^0 \cong S^1$. The fibration is trivial over each hemisphere, so we have $p^{-1}(Z^\pm) \cong D^2 \times S^1$, with $p^{-1}(Z^0) \cong S^1 \times S^1$. So the T-dual also looks like the union of two copies of $D^2 \times S^1$, joined along $S^1 \times S^1$.

However, we have to be careful about the *clutching* that identifies the two copies of $S^1 \times S^1$. In the original Hopf fibration, the clutching function $S^1 \to S^1$ winds once around, with the result that the fundamental group $\mathbb{Z}$ of the fiber T dies in the total space X. But T-duality is supposed to interchange "winding" and "momentum" quantum numbers. So $X^\sharp$ has no winding and is just $S^2 \times S^1$.

So what happened to the clutching function? It shows up in the H-flux of the dual!

6.1.1.2. *T-Duality with a B-field.* To explain this, let's go back to Buscher's derivation of T-duality for the sigma-model with maps $x = (x_1, x_0)\colon \Sigma \to X^\sharp = Z \times S^1$, but this time including the Wess-Zumino term. The action now has the form

$$S(x) = \frac{1}{4\pi\alpha'} \int_\Sigma \left(\|\nabla x_1\|_Z^2 \, d\mathrm{vol}_\Sigma + \frac{R^2}{\alpha'} dx_0 \wedge * dx_0 + x^* B \right) . \tag{6.1}$$

When we dualize the S^1, we have to be careful about the part of B that involves this factor.

In our situation, we are starting with a case where $B^{\pm} = \eta^{\pm} \times d\mathrm{vol}_{S^1}$ is a 2-form over $Z^{\pm} \times S^1$, and $dB^{\pm}$ is a volume form on $Z^{\pm} \times S^1$. Note that $dB^{\pm}$, but not $B^{\pm}$, are supposed to agree on $Z^0 \times S^1$.

In terms of the closed 1-form $\omega = dx_0$, the action becomes

$$S(\omega) = \frac{1}{4\pi\alpha'} \int_{\Sigma} \cdots + \frac{R^2}{\alpha'} \omega \wedge *\omega + \omega \wedge x_1^* \beta,$$

where we've left out terms not involving $x_0 \colon \Sigma \to S^1$, since they don't change under T-duality. As in Section 1.3.2.1, introduce the Lagrange multiplier μ to get

$$S(\omega, \mu) = \frac{1}{4\pi\alpha'} \int_{\Sigma} \cdots + \frac{R^2}{\alpha'} \omega \wedge *\omega + \omega \wedge x_1^* \beta + 2\mu\, d\omega,$$

which if we vary μ gives back the original action. But take the variation in ω instead. We set $\delta S = 0$ and get what we had before but with an extra term:

$$\frac{2R^2}{\alpha'} * \omega + x_1^* \beta + 2d\mu = 0.$$

So $*\omega = \frac{-\alpha'}{R^2}\left(d\mu + \frac{1}{2} x_1^* \beta\right)$ and $\omega = \frac{\alpha'}{R^2} * \left(d\mu + \frac{1}{2} x_1^* \beta\right)$. If $\eta = d\mu$, substituting back into $S(\omega, \mu)$ gives

$$E'(\eta) = \frac{-1}{4\pi\alpha'} \int_{\Sigma} \cdots + \frac{\alpha'}{R^2} \left(\eta + \frac{1}{2} x_1^* \beta\right) \wedge * \left(\eta + \frac{1}{2} x_1^* \beta\right),$$

which has the same form as E except that $R \leftrightarrow \frac{\alpha'}{R}$ and η is *shifted* by $\frac{1}{2} x_1^* \beta$. Recall β is not globally defined; the forms $\beta^{\pm}$ differ by a closed 1-form on Z^0.

6.1.1.3. *K-Theory Matching.* Thus when we apply T-duality starting with $X^\sharp = Z \times S^1$ and the H-flux a generator of $H^3(X^\sharp)$, we see the closed 1-form associated to the T-dual is shifted on one hemisphere relative to the another, the shifting associated to a generator of $H^1(Z^0)$. That shows exactly that the clutching map of the dual theory on X corresponds to the identity map $S^1 \to S^1$, and so the dual spacetime X is not $S^2 \times S^1$ but S^3.

We can also explain this in terms of matching of D-brane charges. If the sigma models on X and $X^\sharp$ are to give indistinguishable physics, the D-brane charges in the two theories must live in isomorphic groups.

Thus we want to require $K^*(X, H) \cong K^{*+1}(X^\sharp, H^\sharp)$. The degree shift comes from interchange of type IIA string theory with type IIB.

Example 6.1 (The Case of $S^2 \times S^1$ and S^3). Let's check this principle of K-theory matching in the case we've been considering, $X = S^3$ fibered by the Hopf fibration over $Z = S^2$. The H-flux on X is trivial, so D-brane changes lie in $K^*(S^3)$, with no twisting. And $K^0(S^3) \cong K^1(S^3) \cong \mathbb{Z}$.

On the T-dual side, we expect to find $X^\sharp = S^2 \times S^1$, also fibered over S^2, but simply by projection onto the first factor. If the H-flux on $X^\sharp$ were trivial, D-brane changes would lie in $K^0(S^2 \times S^1)$ and $K^1(S^2 \times S^1)$, both of which are isomorphic to $\mathbb{Z}^2$, which is *too big*.

On the other hand, we can compute $K^*(S^2 \times S^1, H^\sharp)$ for the class $H^\sharp$ which is k times a generator of $H^3 \cong \mathbb{Z}$, using the Atiyah-Hirzebruch spectral sequence of Section 4.2.3. The differential is

$$H^0(S^2 \times S^1) \xrightarrow{k} H^3(S^2 \times S^1),$$

so when $k = 1$, $K^*(S^2 \times S^1, H^\sharp) \cong K^*(S^3) \cong \mathbb{Z}$ for both $* = 0$ and $* = 1$.

6.1.2. Axiomatics. This discussion suggests we should try to develop an axiomatic treatment of the *topological* aspects of T-duality. Note that we are ignoring many things, such as the underlying metric on spacetime and the auxiliary fields.

6.1.2.1. *Axioms for Topological T-Duality.*

(1) We have a suitable class of spacetimes X each equipped with a principal S^1-bundle $X \to Z$. (X might be required to be a smooth connected manifold.)
(2) For each X, we assume we are free to choose any H-flux $H \in H^3(X, \mathbb{Z})$.
(3) There is an involution (map of period 2, up to equivalence) $(X, H) \mapsto (X^\sharp, H^\sharp)$ keeping the base Z fixed.
(4) If $X = Z \times S^1$ and $H = 0$, then $(X^\sharp, H^\sharp)$ is topologically again $(Z \times S^1, 0)$. If $Z = S^2$ and $(X, H) = (S^3, 0)$, then $(X^\sharp, H^\sharp)$ is as in 6.1.
(5) Naturality: If $\varphi\colon Z_1 \to Z$ and if (X, H) is a pair over Z, then if we form the pull-back diagram

$$\begin{array}{ccc} X_1 & \overset{\widehat{\varphi}}{\dashrightarrow} & X \\ \downarrow & & \downarrow \\ Z_1 & \xrightarrow{\varphi} & Z, \end{array}$$

the T-dual of $(X_1, \widehat{\varphi}^*(H))$ (as a pair over Z_1) is the pull-back of $(X^\sharp, H^\sharp)$ (computed over Z).
(6) $K^*(X, H) \cong K^{*+1}(X^\sharp, H^\sharp)$.

6.2. The Bunke-Schick Construction

Bunke and Schick [**34**] suggested constructing a theory satisfying these axioms by means of a *universal example*. It is known that (for reasonable spaces Z, say CW complexes) all principal S^1-bundles $X \to Z$ come by *pull-back* from a diagram

$$\begin{array}{ccc} X & \dashrightarrow & ES^1 \simeq * \\ \downarrow & & \downarrow \\ Z & \dashrightarrow & B\mathbb{T} \simeq K(\mathbb{Z}, 2) \end{array}$$

Here the map $Z \dashrightarrow K(\mathbb{Z}, 2)$ is unique up to homotopy.

Similarly, every class $H \in H^3(X, \mathbb{Z})$ comes by pull-back from a canonical class via a map $X \dashrightarrow K(\mathbb{Z}, 3)$ unique up to homotopy.

Before we state the Bunke-Schick Theorem, let's begin with a classical theorem from algebraic topology, which will be needed later.

THEOREM 6.2 (Gysin sequence). *Let $X \xrightarrow{p} Z$ be a principal S^1-bundle over a path-connected base Z with Chern class $c \in H^2(Z, \mathbb{Z})$. Then the cohomology groups of X and Z are related by a long exact Gysin sequence*

$$(6.2) \quad \cdots \to H^k(Z, \mathbb{Z}) \xrightarrow{\cup c} H^{k+2}(Z, \mathbb{Z}) \xrightarrow{p^*} H^{k+2}(X, \mathbb{Z}) \xrightarrow{p_!} H^{k+1}(Z, \mathbb{Z}) \xrightarrow{\cup c} H^{k+3}(Z, \mathbb{Z}) \to \cdots .$$

Proof. Proofs can be found in most standard algebraic topology textbooks, but we just indicate how this follows from the Serre spectral sequence

$$H^p(Z, H^q(S^1, \mathbb{Z})) \Rightarrow H^{p+q}(X, \mathbb{Z}),$$

for those familiar with it. The E_2 terms of the spectral sequence look like

$$\begin{array}{cccccc}
q\uparrow & & & & & \\
0 & 0 & 0 & 0 & \cdots & \\
0 & 0 & 0 & 0 & \cdots & \\
H^0(Z) & H^1(Z) & H^2(Z) & H^3(Z) & \cdots & \\
 & \searrow^{d} & \searrow^{d} & \searrow^{d} & & \\
H^0(Z) & H^1(Z) & H^2(Z) & H^3(Z) & \cdots \longrightarrow & p
\end{array},$$

where d is the only nonzero differential, $d_2\colon H^p(Z) \to H^{p+2}(Z)$. Since Z and X are path-connected, $H^0(Z, \mathbb{Z}) \cong \mathbb{Z}$ and has a canonical generator, 1. The class $d(1) \in H^2(Z)$ is c, and since d must have the derivation property, it has to be cup product with c. Then we have exact sequences

$$0 \to E_\infty^{p,1} \to H^p(Z, \mathbb{Z}) \xrightarrow{\cup c} H^{p+2}(Z, \mathbb{Z}) \to E_\infty^{p+2,0} \to 0$$

and

$$0 \to E_\infty^{p,0} \xrightarrow{p^*} H^p(X, \mathbb{Z}) \to E_\infty^{p-1,1} \to 0,$$

from which (6.2) follows. The Gysin map $p_!$, also sometimes called *integration along the fibers*, since that's what it amounts to in terms of de Rham cohomology, can be defined to be the composite

$$H^k(X, \mathbb{Z}) \to E_\infty^{k-1,1} \hookrightarrow H^{k-1}(Z, \mathbb{Z}).$$ □

Theorem 6.3 (Bunke-Schick [**34**]). *There is a classifying space R, unique up to homotopy equivalence, with a fibration*

(6.3)
$$\begin{array}{ccc}
K(\mathbb{Z}, 3) & \longrightarrow & R \\
 & & \downarrow \\
 & & K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2),
\end{array}$$

and any $(X, H) \to Z$ as in the axioms of 6.1.2.1 *comes by pull-back from*

$$\begin{array}{ccc}
X & \dashrightarrow & E \\
\downarrow & & \downarrow \\
Z & \dashrightarrow & R,
\end{array}$$

with the horizontal maps unique up to homotopy and H pulled back from a canonical class in $H^3(E, \mathbb{Z})$.

Furthermore, the k-invariant of the Postnikov tower (6.3) *characterizing R is the cup-product in*

$$H^4(K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2), \mathbb{Z})$$

of the two canonical classes in H^2. The space E in the fibration

$$\begin{array}{ccc} \mathbb{T} & \longrightarrow & E \\ & & \downarrow p \\ & & R \end{array}$$

has the homotopy type of $K(\mathbb{Z},3) \times K(\mathbb{Z},2)$.

Proof. Consider the functor $F\colon Z \rightsquigarrow \{[(X,H)]\}$ assigning to Z the set of all equivalence classes of pairs (X,H) over it, where $X \to Z$ is a principal $\mathbb{T}$-bundle and $H \in H^3(X,\mathbb{Z})$. The functor F takes its values in "pointed sets," since while there is no obvious reason $F(Z)$ must be a group, it certainly has a natural zero element. One can show the functor F must be representable, in the sense that $F(Z)$ is the set of (based) homotopy classes of (based) maps $Z^+ \to R$,[1] where R is some (based) classifying space, by the "abstract nonsense" of the Brown Representability Theorem [**28, 29**]. To get an idea of what R looks like, we can compute $F(S^n)$ by brute force. Clearly $F(S^n) = 0$ for $n < 2$ or for $n > 3$, since a circle bundle over S^n can be nontrivial only if $n = 2$, and the total space of a circle bundle over S^n can have nonzero H^3 only if $n = 2$ or 3. Thus the only non-zero homotopy groups of R can be π_2 and π_3. Since R is simply connected, there is no also difference between based and unbased homotopy classes of maps $S^n \to R$ with $n = 2$ or 3. Furthermore, $\pi_2(R) \cong \mathbb{Z}^2$, since a pair (X,H) over S^2 is classified by two integers, the Chern class of the bundle $X \to S^2$ and the value of H in $H^3(X,\mathbb{Z}) \cong \mathbb{Z}$. Similarly, $\pi_3(R) \cong \mathbb{Z}$, since any pair over S^3 is of the form $(S^3 \times S^1, H \times 1)$, where $H \in H^3(S^3,\mathbb{Z}) \cong \mathbb{Z}$. Since we now know all the homotopy groups of R, Postnikov theory gives a fibration of the form (6.3). The problem is to compute its k-invariant, the class in $H^4(K(\mathbb{Z},2) \times K(\mathbb{Z},2),\mathbb{Z})$ by which (6.3) is pulled back from the universal $K(\mathbb{Z},3)$ fibration

$$\begin{array}{ccc} K(\mathbb{Z},3) = \Omega K(\mathbb{Z},4) & \longrightarrow & \mathrm{pt} \\ & & \downarrow \\ & & K(\mathbb{Z},4). \end{array}$$

Bunke and Schick also give another way of describing R explicitly, which is also useful. Start with the free loop space

$$E = \Lambda K(\mathbb{Z},3) = \mathrm{Map}(S^1, K(\mathbb{Z},3)),$$

on which $\mathbb{T}$ acts by rotating the domain $S^1 \cong \mathbb{T}$. The "Borel construction" gives a homotopy fibration

$$\begin{array}{ccc} E & \xrightarrow{p} & R = E\mathbb{T} \times_{\mathbb{T}} E \\ & & \downarrow c \\ & & B\mathbb{T} = K(\mathbb{Z},2). \end{array} \tag{6.4}$$

[1]Here Z^+ is Z with a disjoint basepoint added; this is simply a device to get around the fact that Z need not have any natural basepoint to begin with.

We can think of c as the Chern class of a circle bundle

$$\begin{array}{ccc} \mathbb{T} & \longrightarrow & E \\ & & \downarrow{\scriptstyle p} \\ & & R. \end{array} \tag{6.5}$$

(Strictly speaking, this is only up to homotopy. To get a genuine circle bundle, replace E by the homotopy-equivalent space $E \times E\mathbb{T}$ with the diagonal action of $\mathbb{T}$, which is now free with R as quotient.) The free loop space comes with a fibration

$$\begin{array}{ccc} \Omega K(\mathbb{Z},3) = K(\mathbb{Z},2) & \longrightarrow & E = \Lambda K(\mathbb{Z},3) \\ & & \downarrow{\scriptstyle e} \\ & & K(\mathbb{Z},3), \end{array}$$

where $\Omega K(\mathbb{Z},3)$ is the *based* loop space and e is evaluation of loops at $1 \in S^1$. Since e has a section, given by constant loops, $E \simeq K(\mathbb{Z},2) \times K(\mathbb{Z},3)$ and we have a canonical class $h \in H^3(E,\mathbb{Z})$ (associated to the map e).

We just need to check that these specific E and R have the right properties. Given a pair (X,H) over a base Z, we have a map $Z \dashrightarrow B\mathbb{T}$ classifying $X \to Z$. This comes from an $\mathbb{T}$-equivariant map $X \overset{\mathbb{T}}{\dashrightarrow} E\mathbb{T}$. We also have a map $X \dashrightarrow K(\mathbb{Z},3)$ classifying H. View this as an equivariant map $X \overset{\mathbb{T}}{\dashrightarrow} \Lambda K(\mathbb{Z},3)$ and take the product to get a commuting diagram

$$\begin{array}{ccc} X & \dashrightarrow & E\mathbb{T} \times E \simeq E \\ \downarrow & & \downarrow{\scriptstyle p} \\ Z & \dashrightarrow & R. \end{array}$$

It's easy to see that this identifies $(X,H) \to Z$ with the pull-back of $(E,h) \xrightarrow{p} R$. From the fibration

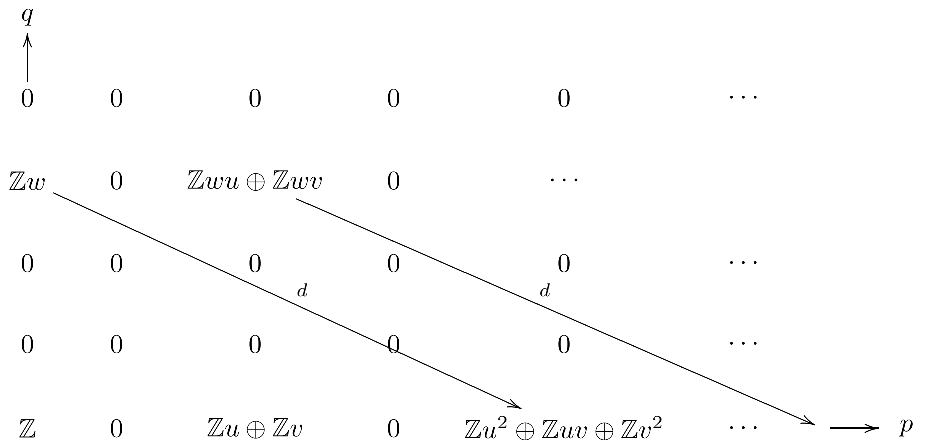
$$\begin{array}{ccc} E & \xrightarrow{p} & R \\ & & \downarrow{\scriptstyle c} \\ & & K(\mathbb{Z},2) \end{array}$$

and the fact that $E \simeq K(\mathbb{Z},2) \times K(\mathbb{Z},3)$, we also see $\pi_2(R) \cong \mathbb{Z}^2$ and $\pi_3(R) \cong \mathbb{Z}$, which gives an independent check of the calculation we did previously.

Now let's go back to the problem of computing the k-invariant. If the k-invariant were trivial, we'd have $R \simeq K(\mathbb{Z},2) \times K(\mathbb{Z},2) \times K(\mathbb{Z},3)$, which would be impossible, since it would imply the H-flux H on X is always the pull-back of a class in $H^3(Z,\mathbb{Z})$, which need not be the case. Think of the Hopf fibration $S^3 = X \xrightarrow{p} Z = S^2$. $H^3(S^3,\mathbb{Z}) \cong \mathbb{Z}$, but only the zero class is pulled back from $H^3(S^2,\mathbb{Z}) = 0$. In fact in this example, $p_!\colon H^3(X,\mathbb{Z}) \to H^2(Z,\mathbb{Z})$ is an isomorphism, and we see that the classes $c_1(p)$ and $p_!(H)$ are independent of one another.

Now recall that $H^*(K(\mathbb{Z},2)\times K(\mathbb{Z},2),\mathbb{Z})$ is a polynomial ring on two generators, both of degree 2. We choose one of these generators to be u, the Chern class of

(6.5), and let w be the canonical generator of $H^3(K(\mathbb{Z},3),\mathbb{Z})$. The other generator v of $H^2(R)$ will be specified shortly. The E_2 term of the Serre spectral sequence of (6.3) has the form



where d sends w to the k-invariant. Since we know this is non-zero, d is injective and $H^3(R,\mathbb{Z})=0$, while the rank of $H^4(R,\mathbb{Z})$ must be 2.

Now consider the Gysin sequence (6.2) applied to the circle bundle (6.5). This has the form

$$0 = H^3(R,\mathbb{Z}) \xrightarrow{p^*} H^3(E,\mathbb{Z}) = \mathbb{Z}w \xrightarrow{p_!} H^2(R,\mathbb{Z}) = \mathbb{Z}u \oplus \mathbb{Z}v$$
$$\xrightarrow{\cdot u} H^4(R,\mathbb{Z}) \xrightarrow{p^*} H^4(E,\mathbb{Z}) = \mathbb{Z}v^2 \xrightarrow{p_!} H^3(R,\mathbb{Z}) = 0.$$

This exact sequence shows that $H^4(R,\mathbb{Z}) \cong \mathbb{Z}^2 \oplus$ (torsion). It also shows that $p_!(w) \in H^2(R,\mathbb{Z})$ is killed under multiplication by u. This class cannot be a multiple of u, since for a pair $(X \xrightarrow{p'} Z, H)$, p' is pulled back from $p\colon E \to R$ and H is pulled back from w, and the example of the Hopf fibration showed that $c_1(p')$ and ${p'}_!(H)$ have to be independent of one another. Furthermore, $p_!(w)$ must be primitive (i.e., cannot be a nontrivial multiple of another class), for otherwise ${p'}_!(H)$ would always be a nontrivial multiple of another class, which is not the case in the Hopf fibration example when H is a generator of $H^3(S^3,\mathbb{Z})$. In particular, $H^4(R,\mathbb{Z})$ is torsion-free. So we let $v = p_!(w)$ and see that u and v are now free generators for $H^2(R,\mathbb{Z})$.

Since uv dies in $H^4(R,\mathbb{Z})$, this means (up to a sign, which is a matter of convention) that the k-invariant has to be uv, and the other generator of $H^4(R,\mathbb{Z})$ is u^2. □

THEOREM 6.4 (Topological T-Duality). *The space R has a homotopy automorphism of period 2, called topological T-duality, that comes from interchanging the two copies of $K(\mathbb{Z},2)$ in (6.3). This gives an involution*

$$[(X \xrightarrow{p} Z, H)] \mapsto [(X^\sharp \xrightarrow{p^\sharp} Z, H^\sharp)]$$

on the space of pairs (X,H) over a base Z, with the property that

$$c_1(p) = (p^\sharp)_!(H^\sharp) \quad \textit{and} \quad c_1(p^\sharp) = (p)_!(H). \tag{6.6}$$

Proof. Because of the fibration (6.3) and the fact that its k-invariant uv is symmetric in the two factors, the "flip" automorphism $\sharp$ interchanging the two factors of $K(\mathbb{Z},2)\times K(\mathbb{Z},2)$ extends to a homotopy automorphism $\sharp$ of R of period 2, and thus to an involutive natural transformation from the functor $F\colon Z \rightsquigarrow \{[(X,H)]\}$ to itself. It remains to verify the stated relationship (6.6) between characteristic classes involving the Gysin map.

We saw from the proof of Theorem 6.3 that the two generators of $H^2(R,\mathbb{Z})$ can be taken to be $u=c_1(p)$ and $v=p_!(w)$, where w is the canonical generator of $H^3(E,\mathbb{Z})$ and p is as in (6.5). These are interchanged under $\sharp$, so (6.6) follows. $\square$

The treatment above shows that it is possible to satisfy all the T-duality axioms of Section 6.1.2.1 above, except perhaps for the last one dealing with twisted K-theory. We will come back to this axiom in the next chapter.

Example 6.5 (The Case of $S^2\times S^1$ and S^3, Revisited). We conclude this chapter by looking again at the case of $S^2\times S^1$ and S^3 from the beginning of this chapter. Let $a\in H^2(S^2)$, $b\in H^1(S^1)$ be the usual generators. Look at the diagram

$$\begin{array}{ccc} (X,H)=(S^3,0) & & (X^\sharp,H^\sharp)=(S^2\times S^1,a\times b) \\ & \searrow^{p} \quad {}^{p^\sharp}\swarrow & \\ & Z & \end{array}$$

We have $c_1(p)=(p^\sharp)_!(H^\sharp)=a$, $c_1(p^\sharp)=p_!(H)=0$. So indeed T-duality interchanges

$$c_1(p)\leftrightarrow (p^\sharp)_!(H^\sharp),\quad c_1(p^\sharp)\leftrightarrow p_!(H).$$