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Number 113

Families of Riemann Surfaces and Weil-Petersson Geometry

Scott A. Wolpert



American Mathematical Society
with support from the
National Science Foundation



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by the

American Mathematical Society

Providence, Rhode Island

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National Science Foundation



NSF-CBMS Regional Research Conferences in the Mathematical Sciences
on Families of Riemann Surfaces and Weil-Petersson Geometry
held at Central Connecticut State University, New Britain, CT
July 20–24, 2009

Partially supported by the National Science Foundation.
The author acknowledges support from the Conference Board of
the Mathematical Sciences and NSF Grant DMS 0834134.

2000 *Mathematics Subject Classification*. Primary 20F67, 30F60, 32G15, 37F30;
Secondary 11F41, 14H15, 32Q05, 32Q45.

For additional information and updates on this book, visit
www.ams.org/bookpages/cbms-113

Library of Congress Cataloging-in-Publication Data

Wolpert, Scott A.

Families of Riemann surfaces and Weil-Petersson geometry / Scott A. Wolpert.

p. cm. – (CBMS regional conference series in mathematics ; no. 113)

Includes bibliographical references and index.

ISBN 978-0-8218-4986-6 (alk. paper)

1. Riemann surfaces. 2. Teichmüller spaces. 3. Hyperbolic groups. 4. Ergodic theory.
5. Geometry, Riemannian. I. Title.

QA337.W65 2010

515'.93-dc22

2010011413

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Printed in the United States of America.

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10 9 8 7 6 5 4 3 2 1 15 14 13 12 11 10

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Preface

These written lectures are the companion to the NSF-CBMS Regional Research Conference in the Mathematical Sciences, organized July 20-24, 2009 at Central Connecticut State University, by Eran Makover and Jeffrey McGowan. My goal for the lectures is a generally self-contained course for graduate students and postgraduates. The topics run across current research areas. By plan the approach is didactic. Concepts are developed across multiple lectures. Opportunities are taken to introduce general concepts, to present recurring methods and to generally provide proofs. Guides to the research literature are included.

The study of Riemann surfaces continues to be an interface for algebra, analysis, geometry and topology. I hope that in part I am able to suggest the interaction to the audience and reader. The lectures are not intended as a proper research summary or history of the field. A collection of current and important topics are not included. Material is not always presented following the historical development of concepts. The references are not all inclusive but are intended only as a lead-in to the literature. Further readings are provided at the ends of chapters.

I thank the Conference Board of the Mathematical Sciences and the National Science Foundation for supporting the undertaking. NSF Grant DMS 0834134 supported the Regional Conference and lectures. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

First and foremost, I would like to thank Eran Makover and Jeffrey McGowan. The idea for a conference, the planning and all the arrangements were smoothly and efficiently handled by Eran and Jeff. On behalf of the participants I would like to express appreciation to Central Connecticut State University for providing arrangements and facilities. I am most appreciative for the conference participants' engagement and feedback. Also, I especially would like to thank Bill Goldman, Zheng (Zeno) Huang, Zachary McGuirk, Babak Modami, Kunio Obitsu, Athanase Papadopoulos and Mike Wolf for detailed comments, feedback and contributions.

Scott A. Wolpert
January, 2010
College Park, Maryland

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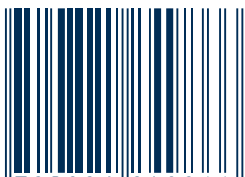
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