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Number 115

The Mutually Beneficial Relationship of Graphs and Matrices

Richard A. Brualdi

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with support from the
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Dedicated to my Grandchildren:

Samantha Brualdi Shalgos
Andrew Brualdi Shalgos
Olga Brualdi

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Preface

This monograph is based on the ten lectures I gave at Iowa State University in Ames during the week of July 12-16, 2010. The purpose of the lectures was to show the fascinating and mutually beneficial relationship between matrices and graphs (the nonzero pattern of a matrix): (i) knowledge about one of the graphs that can be associated with a matrix is used to illuminate matrix properties and to get better information about the matrix, and (ii) linear algebraic properties of one of the matrices associated with a graph is used to get useful combinatorial information about the graph. The lectures were not intended to be comprehensive on any of the topics treated; they could not have been within the time framework imposed by ten one-hour lectures. Nor were the lectures intended to cover all instances in which the interplay between matrices and graphs has turned out to be useful; again an impossibility within the time framework. The particular content of the lectures was chosen for its accessibility, beauty, and current relevance, and for the possibility of enticing the audience to want to learn more. It was, of course, influenced by the author's personal interests and expertise.

In this monograph, I have stayed within the context of the lectures, and have avoided writing a more comprehensive book. In most cases I have not given original references for results if they are readily available in one or more books referenced. Just as we did for the lectures, we assume that the reader is familiar with many of the basic concepts and facts of matrix theory and graph theory. We define some standard terms but many are presumed known and can be found in most elementary and advanced books.

I am indebted to Leslie Hogben and Bryan Shader for organizing this CBMS Regional Conference and for suggesting me as principal lecturer. They did a superb job, from recruiting a diverse group of participants to arranging a stimulating and fun daily schedule with afternoon and evening activities. I would also like to express my gratitude to the participants for their attention, stimulating questions, and camaraderie. Finally, I want to thank the Department of Mathematics of Iowa State University for hosting the conference and the National Science Foundation under grant number DMS 0938261 for financially supporting it.

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Graphs and matrices enjoy a fascinating and mutually beneficial relationship. This interplay has benefited both graph theory and linear algebra. In one direction, knowledge about one of the graphs that can be associated with a matrix can be used to illuminate matrix properties and to get better information about the matrix. Examples include the use of digraphs to obtain strong results on diagonal dominance and eigenvalue inclusion regions and the use of the Rado-Hall theorem to deduce properties of special classes of matrices. Going the other way, linear algebraic properties of one of the matrices associated with a graph can be used to obtain useful combinatorial information about the graph. The adjacency matrix and the Laplacian matrix are two well-known matrices associated to a graph, and their eigenvalues encode important information about the graph. Another important linear algebraic invariant associated with a graph is the Colin de Verdière number, which, for instance, characterizes certain topological properties of the graph.

This book is not a comprehensive study of graphs and matrices. The particular content of the lectures was chosen for its accessibility, beauty, and current relevance, and for the possibility of enticing the audience to want to learn more.

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