The Mutually Beneficial Relationship of Graphs and Matrices

Richard A. Brualdi
The Mutually Beneficial Relationship of Graphs and Matrices
The Mutually Beneficial Relationship of Graphs and Matrices

Richard A. Brualdi
Dedicated to my Grandchildren:
Samantha Brualdi Shalgos
Andrew Brualdi Shalgos
Olga Brualdi
Contents

Preface ix

Chapter 1. Some Fundamentals 1
  1.1. Nonnegative Matrices 1
  1.2. Symmetric Matrices 7
  Bibliography 11

Chapter 2. Eigenvalues of Graphs 13
  2.1. Some Basic Properties 13
  2.2. Eigenvalues and Graph Parameters 16
  2.3. Graphs with small $\lambda_{\text{max}}$ 19
  2.4. Laplacian Matrix of a Graph 21
  Bibliography 23

Chapter 3. Rado-Hall Theorem and Applications 25
  3.1. Rado-Hall Theorem 25
  3.2. Applications 27
  Bibliography 31

Chapter 4. Colin de Verdière Number 33
  4.1. Motivation and Definition 33
  4.2. Colin de Verdière Number and Graph Properties 35
  Bibliography 38

Chapter 5. Classes of Matrices of Zeros and Ones 39
  5.1. Equivalent Formulations 39
  5.2. The Classes $A(R, S)$ 40
  5.3. A Generalization 45
  Bibliography 47

Chapter 6. Matrix Sign Patterns 49
  6.1. Sign-Nonsingular Matrices 49
  6.2. An Application 53
  6.3. Spectrally Arbitrary Sign Patterns 55
  Bibliography 57

Chapter 7. Eigenvalue Inclusion and Diagonal Products 59
  7.1. Some Classical Theorems 59
  7.2. Diagonal Products and Nonsingularity 62
Bibliography 66

Chapter 8. Tournaments 67
  8.1. Landau’s Theorem 67
  8.2. A Special Tournament in $T(R)$ 71
  8.3. Eigenvalues of Tournament Matrices 74
Bibliography 76

Chapter 9. Two Matrix Polytopes 77
  9.1. The Doubly Stochastic Polytope 77
  9.2. Alternating Sign Matrices 79
  9.3. The Alternating Sign Matrix Polytope 81
  9.4. ASM Patterns 83
Bibliography 86

Chapter 10. Digraphs and Eigenvalues of $(0,1)$-matrices 87
  10.1. $(0,1)$-matrices with all Eigenvalues Positive 87
  10.2. Totally Nonnegative Matrices 90
  10.3. Totally Nonnegative $(0,1)$-matrices 91
Bibliography 93

Index 95
Preface

This monograph is based on the ten lectures I gave at Iowa State University in Ames during the week of July 12-16, 2010. The purpose of the lectures was to show the fascinating and mutually beneficial relationship between matrices and graphs (the nonzero pattern of a matrix): (i) knowledge about one of the graphs that can be associated with a matrix is used to illuminate matrix properties and to get better information about the matrix, and (ii) linear algebraic properties of one of the matrices associated with a graph is used to get useful combinatorial information about the graph. The lectures were not intended to be comprehensive on any of the topics treated; they could not have been within the time framework imposed by ten one-hour lectures. Nor were the lectures intended to cover all instances in which the interplay between matrices and graphs has turned out to be useful; again an impossibility within the time framework. The particular content of the lectures was chosen for its accessibility, beauty, and current relevance, and for the possibility of enticing the audience to want to learn more. It was, of course, influenced by the author’s personal interests and expertise.

In this monograph, I have stayed within the context of the lectures, and have avoided writing a more comprehensive book. In most cases I have not given original references for results if they are readily available in one or more books referenced. Just as we did for the lectures, we assume that the reader is familiar with many of the basic concepts and facts of matrix theory and graph theory. We define some standard terms but many are presumed known and can be found in most elementary and advanced books.

I am indebted to Leslie Hogben and Bryan Shader for organizing this CBMS Regional Conference and for suggesting me as principal lecturer. They did a superb job, from recruiting a diverse group of participants to arranging a stimulating and fun daily schedule with afternoon and evening activities. I would also like to express my gratitude to the participants for their attention, stimulating questions, and camaraderie. Finally, I want to thank the Department of Mathematics of Iowa State University for hosting the conference and the National Science Foundation under grant number DMS 0938261 for financially supporting it.
Index

alternating sign matrix (ASM), 79
  pattern, 83

bipartite graph, 1, 78
  complete, 1
  eigenvalues, 15
  left vertex, 1
  right vertex, 1

diagonal product, 49, 61–64
  signed, 49
digraph, 49, 60, 87
  acyclic, 88
  cycle, 51
  eigenvalues, 87
    nonnegative, 88
    positive, 88
  strongly connected, 60
  weighted, 49
dimer problem, 53
equal union property, 54

graph, 13
  adjacency matrix, 13
    weighted, 33
  algebraic connectivity, 22
  alternating sign, 85
  biclique, 27
  biclique partition number, 27
    lower bound, 29
  chromatic number, 16
    lower bound, 18
    upper bound, 18
  clique, 17, 27
  Colin de Verdière number, 33–35
    complete graph, 35
    linklessly embeddable, 36
    outerplanar, 36
    path, 36
    planar, 36, 37
  complement, 17, 35
  connected, 14
  diameter, 15
  edge connectivity, 22
  eigenvalues, 13, 14
  independence number, 16
  Laplacian matrix, 21
  minor, 35
    forbidden, 35
    minor-closed, 35
    proper, 35
  multicolored, 30
  spanning forest, 30
  spanning tree, 30
  planar, 35
  regular, 17
  Smith, 20
  spanning tree
    number, 21
    vertex connectivity, 22
  interchange, 44
    generalization, 47
  graph, 44
  L-matrix, 53
  Laplacian matrix, 21
  lemniscate, 62
  majorization, 42, 45, 69, 73
    Muirhead’s lemma, 45
  matrix
    absolute value, 61
    alternating sign, 79
      conjecture, 80
    ASM, 79
    biadjacency, 39
    classes, 39, 45, 69, 85
    co-rank, 34
    determinant, 21, 49
    diagonally dominant, 59, 63
matrix (cont.)
  double staircase pattern, 91
  doubly stochastic, 77
  incidence, 39, 54
  irreducible, 6, 14, 60, 61
  L-matrix, 53
  permanent, 49, 52
  reducible, 6
  sign pattern, 50, 55
    potentially nilpotent, 56
    SAP, 55
    sign-nonsingular, 50
    SNS, 50, 51
    spectrally arbitrary, 55
  signing, 52
  simultaneous permutation, 2
  skew-symmetric, 28
    eigenvalues, 28
  symmetric, 7
    interlacing, 8
  TNN, 90
  totally nonnegative, 90
    characterization, 92
  totally positive, 90
  weakly diagonally dominant, 59, 61
matroid, 25, 43, 70
  basis, 25
  circuit, 25
  dependent set, 25
  independent set, 25
  rank, 25
  rank function, 25
    submodular inequality, 26
  uniform, 25
minor monotone, 35

optimal assignment problem (OAP), 63
duality theorem, 64

Perron-Frobenius theory, 3
polytope
  alternating sign matrix, 81
  extreme points, 83
  doubly stochastic, 78
  extreme points, 78

qualitative class, 50

SDR, 26, 55
  independent, 26, 43
  spectral radius, 1, 2
square ice, 80
Strong Arnold Pproperty, 34

theorem
  Ao-Hanson-Guidili-
    Gárfás-Thomassé-Weidl, 72
  Bassett-Maybee-Quirk, 51
  Birkhoff, 78
  Brauer-Gentry, 74
  Brualdi-Kirkland, 88
  Camion-Hoffman, 65
  Colin de Verdière, 35
  de Boor-Pinkus, 91
  Gale-Ryser, 41, 70, 72, 73
  Geršgorin, 60
  Graham-Pollak, 27
  Kastelyn, 53
  Kuratowski, 35
  Landau, 69, 73
  Lindström-Tverberg, 54
  McKay-Oggier-Royle-Sloane-Wanless-
    Wilf, 88
  Rado, 42
  Rado-Hall, 26, 27, 55, 70, 78
  Robertson-Seymour, 35
  Schwarz, 3
  Striker-Behrend-Knight, 81
  Taussky, 61
  Zeilberger, 80
tournament, 67
  loss, 67
  loss vector, 67
  nearly regular, 74
  regular, 74
  score, 67
  score vector, 67
  transitive, 69, 71, 72
tournament matrix, 68
eigenvalues, 74
  spectral radius, 75
transfer, 45

vector
  support, 36
    negative, 36
    positive, 36
Titles in This Series

115 Richard A. Brualdi, The mutually beneficial relationship of graphs and matrices, 2011
114 Mark Gross, Tropical geometry and mirror symmetry, 2011
113 Scott A. Wolpert, Families of Riemann surfaces and Weil-Petersson geometry, 2010
112 Zhongshan Wang, Topological quantum computation, 2010
111 Jonathan Rosenberg, Topology, $C^*$-algebras, and string duality, 2009
110 David Nualart, Malliavin calculus and its applications, 2009
109 Robert J. Zimmer and Dave Witte Morris, Ergodic theory, groups, and geometry, 2008
108 Alexander Koldobsky and Vladyslav Yaskin, The interface between convex geometry and harmonic analysis, 2008
107 Fan Chung and Linyuan Lu, Complex graphs and networks, 2006
106 Terence Tao, Nonlinear dispersive equations: Local and global analysis, 2006
105 Christoph Thiele, Wave packet analysis, 2006
104 Donald G. Saari, Collisions, rings, and other Newtonian $N$-body problems, 2005
103 Iain Raeburn, Graph algebras, 2005
102 Ken Ono, The web of modularity: Arithmetic of the coefficients of modular forms and $q$ series, 2004
101 Henri Darmon, Rational points on modular elliptic curves, 2004
100 Alexander Volberg, Calderón-Zygmund capacities and operators on nonhomogeneous spaces, 2003
99 Alain Lascoux, Symmetric functions and combinatorial operators on polynomials, 2003
98 Alexander Varchenko, Special functions, KZ type equations, and representation theory, 2003
97 Bernd Sturmfels, Solving systems of polynomial equations, 2002
96 Niky Kamran, Selected topics in the geometrical study of differential equations, 2002
95 Benjamin Weiss, Single orbit dynamics, 2000
94 David J. Saltman, Lectures on division algebras, 1999
93 Goro Shimura, Euler products and Eisenstein series, 1997
91 J. P. May et al., Equivariant homotopy and cohomology theory, dedicated to the memory of Robert J. Piacenza, 1996
90 John Roe, Index theory, coarse geometry, and topology of manifolds, 1996
89 Clifford Henry Taubes, Metrics, connections and gluing theorems, 1996
88 Craig Huneke, Tight closure and its applications, 1996
87 John Erik Fornæss, Dynamics in several complex variables, 1996
86 Sorin Popa, Classification of subfactors and their endomorphisms, 1995
85 Michio Jimbo and Tetsuji Miwa, Algebraic analysis of solvable lattice models, 1994
84 Hugh L. Montgomery, Ten lectures on the interface between analytic number theory and harmonic analysis, 1994
83 Carlos E. Kenig, Harmonic analysis techniques for second order elliptic boundary value problems, 1994
82 Susan Montgomery, Hopf algebras and their actions on rings, 1993
81 Steven G. Krantz, Geometric analysis and function spaces, 1993
80 Vaughan F. R. Jones, Subfactors and knots, 1991
78 Edward Formanek, The polynomial identities and variants of $n \times n$ matrices, 1991
77 Michael Christ, Lectures on singular integral operators, 1990
76 Klaus Schmidt, Algebraic ideas in ergodic theory, 1990
75 F. Thomas Farrell and L. Edwin Jones, Classical aspherical manifolds, 1990
74 Lawrence C. Evans,  Weak convergence methods for nonlinear partial differential equations, 1990
73 Walter A. Strauss,  Nonlinear wave equations, 1989
72 Peter Orlik,  Introduction to arrangements, 1989
71 Harry Dym,  $J$ contractive matrix functions, reproducing kernel Hilbert spaces and interpolation, 1989
70 Richard F. Gundy,  Some topics in probability and analysis, 1989
69 Frank D. Grosshans, Gian-Carlo Rota, and Joel A. Stein,  Invariant theory and superalgebras, 1987
68 J. William Helton, Joseph A. Ball, Charles R. Johnson, and John N. Palmer,  Operator theory, analytic functions, matrices, and electrical engineering, 1987
67 Harald Upmeier,  Jordan algebras in analysis, operator theory, and quantum mechanics, 1987
66 G. Andrews,  $q$-Series: Their development and application in analysis, number theory, combinatorics, physics and computer algebra, 1986
65 Paul H. Rabinowitz,  Minimax methods in critical point theory with applications to differential equations, 1986
64 Donald S. Passman,  Group rings, crossed products and Galois theory, 1986
63 Walter Rudin,  New constructions of functions holomorphic in the unit ball of $C^n$, 1986
62 Béla Bollobás,  Extremal graph theory with emphasis on probabilistic methods, 1986
61 Mogens Flensted-Jensen,  Analysis on non-Riemannian symmetric spaces, 1986
60 Gilles Pisier,  Factorization of linear operators and geometry of Banach spaces, 1986
59 Roger Howe and Allen Moy,  Harish-Chandra homomorphisms for $p$-adic groups, 1985
58 H. Blaine Lawson, Jr.,  The theory of gauge fields in four dimensions, 1985
57 Jerry L. Kazdan,  Prescribing the curvature of a Riemannian manifold, 1985
56 Hari Bercovici, Ciprian Foiaş, and Carl Pearcy,  Dual algebras with applications to invariant subspaces and dilation theory, 1985
55 William Arveson,  Ten lectures on operator algebras, 1984
54 William Fulton,  Introduction to intersection theory in algebraic geometry, 1984
53 Wilhelm Klingenberg,  Closed geodesics on Riemannian manifolds, 1983
52 Tsit-Yuen Lam,  Orderings, valuations and quadratic forms, 1983
51 Masamichi Takesaki,  Structure of factors and automorphism groups, 1983
50 James Eells and Luc Lemaire,  Selected topics in harmonic maps, 1983
49 John M. Franks,  Homology and dynamical systems, 1982
48 W. Stephen Wilson,  Brown-Peterson homology: an introduction and sampler, 1982
47 Jack K. Hale,  Topics in dynamic bifurcation theory, 1981
45 Ronald L. Graham,  Rudiments of Ramsey theory, 1981
44 Phillip A. Griffiths,  An introduction to the theory of special divisors on algebraic curves, 1980
43 William Jaco,  Lectures on three-manifold topology, 1980
42 Jean Dieudonné,  Special functions and linear representations of Lie groups, 1980
41 D. J. Newman,  Approximation with rational functions, 1979
40 Jean Mawhin,  Topological degree methods in nonlinear boundary value problems, 1979
39 George Lusztig,  Representations of finite Chevalley groups, 1978
38 Charles Conley,  Isolated invariant sets and the Morse index, 1978
37 Masayoshi Nagata,  Polynomial rings and affine spaces, 1978

For a complete list of titles in this series, visit the AMS Bookstore at www.ams.org/bookstore/.
Graphs and matrices enjoy a fascinating and mutually beneficial relationship. This interplay has benefited both graph theory and linear algebra. In one direction, knowledge about one of the graphs that can be associated with a matrix can be used to illuminate matrix properties and to get better information about the matrix. Examples include the use of digraphs to obtain strong results on diagonal dominance and eigenvalue inclusion regions and the use of the Rado-Hall theorem to deduce properties of special classes of matrices. Going the other way, linear algebraic properties of one of the matrices associated with a graph can be used to obtain useful combinatorial information about the graph. The adjacency matrix and the Laplacian matrix are two well-known matrices associated to a graph, and their eigenvalues encode important information about the graph. Another important linear algebraic invariant associated with a graph is the Colin de Verdière number, which, for instance, characterizes certain topological properties of the graph.

This book is not a comprehensive study of graphs and matrices. The particular content of the lectures was chosen for its accessibility, beauty, and current relevance, and for the possibility of enticing the audience to want to learn more.