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Daniel T. Wise

American Mathematical Society
with support from the
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http://dx.doi.org/10.1090/cbms/117

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Published for the
Conference Board of the Mathematical Sciences
by the
American Mathematical Society
Providence, Rhode Island
with support from the
National Science Foundation

NSF/CBMS Regional Research Conference
on 3-Manifolds, Artin Groups, and Cubical Geometry
held at City University of New York, August 1–5, 2011.

Partially supported by the National Science Foundation Grant 1040900.

Research supported by NSERC.

2010 *Mathematics Subject Classification.* Primary 20F67, 20F06, 57M99, 20E26.

Author photo courtesy of Yael Halevi-Wise.

For additional information and updates on this book, visit
www.ams.org/bookpages/cbms-117

Library of Congress Cataloging-in-Publication Data

Wise, Daniel T., 1971–
From riches to raags : 3-manifolds, right-angled artin groups, and cubical geometry / Daniel T. Wise.
p. cm. — (CBMS Regional conference series in mathematics ; number 117)
Includes bibliographical references and index.
ISBN 978-0-8218-8800-1 (alk. paper)
1. Hyperbolic groups. 2. Group theory. I. Title.

QA171.W735 2011
512′.2—dc23 2012032056

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10 9 8 7 6 5 4 3 2 1 17 16 15 14 13 12

To Yael

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Acknowledgement

I am enormously grateful to Jason Behrstock for organizing the NSF-CBMS conference together with Abhijit Champanerkar. I am also grateful to the conference participants for their feedback, to the CUNY Graduate Center for hosting the conference, the CBMS for choosing this topic, and the NSF for funding it. My research has been supported by NSERC and undertaken at McGill University.

During my 2008-2009 sabbatical at the Hebrew University I benefited tremendously from the feedback and encouragement of Zlil Sela, with whom I went through the project on *The Structure of Groups with a Quasiconvex Hierarchy* that led to this conference. Valuable criticism and helpful corrections were subsequently provided by many other friends and colleagues, among them Ian Agol, Nicolas Bergeron, Hadi Bigdely, Collin Bleak, David Gabai, Mark Hagen, Jason Manning, Eduardo Martinez-Pedroza, Daniel Moskovitch, Piotr Przytycki, and Eric Swenson. I am grateful to Martin Bridson for initiating my interest in nonpositively curved square complexes. My daughter Talia helped with the figures and my parents Batya and Michael helped me with crayons and fractions respectively, and have been a source of continuing and enthusiastic support.

It is a great fortune to have wonderful collaborators who spent much time and energy working with me on the ideas presented here: Tim Hsu on linearity, separability in right-angled Artin groups, and cubulating malnormal amalgams; Chris Hruska on the finiteness properties of the dual cube complex, the bounded packing property, and the tower approach to the spelling theorem for one-relator groups with torsion. I've engaged with Michah Sageev's seminal work on CAT(0) cube complexes for the last 10 years and have learned many things working with him. My long-term collaboration with Frédéric Haglund on special cube complexes is at the heart of the matter here, and I am additionally building on his influential earlier ideas.

Finally, I am grateful to my adorable and loving wife Yael for giving me the time to write these notes, and for serving as a partner, a cheerleader, and a role model in all parts of my life.

Note to the reader

The target audience for these lecture notes consists of younger researchers at the beginning of their careers as well as seasoned researchers in neighboring areas interested in quickly acquiring the viewpoint. The requisite background for reading this text should be at the level of an introductory course on geometric group theory or even just hyperbolic groups, though some comfort with graphs of groups would be helpful. I have attempted to include all defined terms and notation in an index at the end of the document, and hope that the intrepid reader will be able to dive into a chapter of interest and work outwards.

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ISBN 978-0-8218-8800-1

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